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MINIMAX AND H^{∞}-OPTIMAL CONTROL OF LINEAR UNSTEADY SYSTEMS

O. Lobok, B. Goncharenko
National University of Food Technologies
L. Vihrova
Central Ukrainian National Technical University

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Corresponding author:
O. Lobok
E-mail:
npnuht@ukr.net

ABSTRACT

The problem of synthesis of minimax control for the dynamic, described by the system of differential equations (taking into account the state, controls, perturbations and initial conditions, with the given equation of observation inclusive) of objects functioning in accordance with the integral-quadratic quality criterion in uncertainty is solved in the work.

External perturbations, errors, and initial conditions were assumed to belong to a number of uncertainties. The task of finding optimal control in the form of a feedback object that minimizes the performance criterion is presented in the form of a minimum maximal uncertainty control problem. In the absence of ready-made solution paths, this problem is reduced to a H^{∞}-control problem under the most unfavorable disturbances, and in addition to a dynamic game problem with zero sum and a certain price for the game, and a strategy for solving it is proposed that offers a way to new results.

The problem of finding the optimal control and the initial state that maximize the quality criterion is considered in the framework of the optimization problem solved by the Lagrange multiplier method after introducing the auxiliary scalar function, the Hamiltonian. It is shown that to find the maximum value of the criterion, either the necessary condition of the extremum of the first kind can be used, which depends on the ratio of the first variation of the criterion and the first variations of the control vectors and the initial state, or also the necessary condition of the extremum of the second kind, which depends on the sign of the second variation. For the first and second variations, formulas are given that can be used for calculations.

It is suggested to solve the control search problem in two steps: search for an intermediate solution at fixed values of control vectors and errors, and then search for final optimal control. Consideration is also given to solving H^{∞}-optimal control for infinite control time with respect to the signal from the compensator output, as well as solving the corresponding Riccati matrix algebraic equations.

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МІНІМАКСНЕ ТА H^∞-ОПТИМАЛЬНЕ КЕРУВАННЯ ЛІНІЙНИМИ НЕСТАЦІОНАРНИМИ СИСТЕМАМИ

О. П. Лобок, Б. М. Гончаренко
Національний університет харчових технологій
Л. Г. Віхрова
Центральноукраїнський національний технічний університет

У статті розв'язано задачу синтезу мінімаксного керування для динамічних, описаних системою диференційних рівнянь (з урахуванням стану, керувань, збурень і початкових умов, з наведеним рівнянням спостереження включно), об'єктів, що функціонують з інтегрально-квадратичним критерієм якості в умовах невизначеності.

Припускалося, що зовнішні збурення, похибки та початкові умови належать певній множині невизначеностей. Задача пошуку оптимального керування у вигляді зворотного по виходу об'єкта зв'язку, який мінімізує критерій функціонування, представлена у вигляді мінімаксної задачі оптимального керування за умов невизначеностей. За відсутності готових шляхів розв'язання показане зведення цієї задачі до задачі H^∞-керування при найбільш несприятливих збуреннях і, крім того, до динамічної ігрової задачі з нульовою сумою та визначеною ціною гри, наведена стратегія її розв'язання, що пропонує шлях до нових результатів.

Завдання пошуку оптимального керування і початкового стану, що максимізують критерій якості, розглянуто в рамках оптимізаційної задачі, яку розв'язано методом множників Лагранжа після введення допоміжної скалярної функції — гамільтоніана. Показано, що для знаходження максимального значення критерію може бути використана або необхідна умова екстремуму першого роду, що залежить від співвідношення першої варіації критерію та перших варіацій векторів керування і початкового стану, або необхідна умова екстремуму другого роду, що залежить від знака другої варіації. Для перших і других варіацій наведено формули, які можуть використовуватися для розрахунків.

Запропоновано задачу пошуку керування розв'язувати в два етапи: пошук проміжного розв'язку при фіксованих значеннях векторів керування та похибки і наступний пошук остаточного оптимального керування. Розглянуто також розв'язання H^∞-оптимального керування на нескінченному часі з урахуванням сигналу з виходу компенсатора, а також розв'язання відповідних матричних алгебраїчних рівнянь типу Рікатті.

Ключові слова: *мінімаксне керування, робастні регулятори, системи з невизначеностями, оптимізація, матрична форма.*

Formulation of the problem. Initially, the main results of studies of linear automatic control systems were the notion of stability and its criteria based on characteristic polynomials. Subsequently, with the development of radio engineering and electronic automation systems, frequency research methods, which later expanded to impulse, discrete, and nonlinear systems in connection with the development of computing, became the main ones. The advancement of astronautics has led to the study of automatic systems in the state of space, the idea of optimizing control systems with the simultaneous optimization of their quality indicators.

Subsequent progress has made it possible to combine frequency with methods of state space research, which in addition to optimization has made it possible to solve problems with any uncertainties — robust control. However, the uncertainty of the frequency response of control objects is limited in the — norm and can be specified in both parametric and matrix form when described in the state space [1].

For uncertainties, it is fruitful to apply a minimax approach when the optimal controller is in the state of the object, which operates under uncertainty so that it minimizes the maximum error (deviation of the current state of the system from a given or desired one) from the set of possible values taking into account the most unfavorable perturbations that can affect an object or system. However, the solution to this problem is not always obvious [2], and its search requires more research.

Consider a dynamic object described by the following system of differential equations [3]

$$\begin{cases} \dfrac{dx(t)}{dt} = A(t)x(t) + B(t)u(t) + F_w(t)w(t), \qquad t_0 < t < T, \\ x(t_0) = F_0 x_0, \end{cases} \tag{1}$$

where $x(t) \in R^{n_x}$ — state vector, $u(t) \in R^{n_u}$ — control vector, $w(t) \in R^{n_w}$ — unknown vector of external perturbations acting on an object, $x_0 \in R^{n_0}$ — unknown vector of initial conditions, $A(t) \in R^{n_x \times n_x}$, $B(t) \in R^{n_x \times n_u}$, $F_w(t) \in R^{n_x \times n_w}$, $F_0 \in R^{n_x \times n_0}$ — given matrices of corresponding dimensions.

Let the object be monitored by the equation

$$y(t) = C(t)x(t) + F_v(t)v(t), \tag{2}$$

where $y(t) \in R^{n_y}$ — the result of observation, $v(t) \in R^{n_v}$ — unknown measurement errors, $C(t) \in R^{n_y \times n_x}$, $F_v(t) \in R^{n_y \times n_v}$ — known matrices.

Consider and choose the integral-quadratic criterion of quality of functioning of the object in the form

$$I(u) = \int_{t_0}^{T} \left(x^T(t)G_x(t)x(t) + u^T(t)G_u(t)u(t) \right) dt + x^T(T)G_f x(T), \tag{3}$$

where $G_x(t) \in R^{n_x \times n_x}$, $G_u(t) \in R^{n_u \times n_u}$, $G_f \in R^{n_x \times n_x}$ — given symmetric weight matrices, and they are assumed to satisfy the conditions $G_x(t) = G_x^T(t) \geq 0$, $G_u(t) = G_u^T(t) > 0$, $G_f = G_f^T \geq 0$.

Here "T"— means the operation of transposing the matrix, $G = G^T$ — means that the matrix is symmetric, $G > 0$ ($G \geq 0$) — the condition of positive (inalienable) definiteness of the matrix, i.e. the matrix G has positive or inalienable eigenvalues.

With respect to the unknown vector of external perturbations $w(t)$, the vector of measurement errors $v(t)$ and the vector of initial conditions x_0, they are assumed to belong to the next set of permissible perturbations (uncertainties)

$$\Omega_\xi = \left\{ \begin{array}{l} \xi:\ \xi = (w(t), v(t), x_0),\ w(t) \in L_2(t_0, T), \\ v(t) \in L_2(t_0, T),\ x_0 \in R^{n_0};\ \|\xi\|^2 \leq 1 \end{array} \right\}, \tag{4}$$

where the norm $\|\xi\|$ of a vector-valued function ξ is defined by the following expression

$$\|\xi\|^2 = \int_{t_0}^{T} \left(w^T(t) R_w(t) w(t) + v^T(t) R_v(t) v(t) \right) dt + \left(x_0 - \hat{x}_0 \right)^T R_0 \left(x_0 - \hat{x}_0 \right), \tag{5}$$

in which $R_w(t) \in R^{n_w \times n_w}$, $R_v(t) \in R^{n_v \times n_v}$, $R_0 \in R^{n_0 \times n_0}$ — are given weight matrices, and $R_w(t) = R_w^T(t) \geq 0$, $R_v(t) = R_v^T(t) > 0$, $R_0 = R_0^T \geq 0$, $\hat{x}_0 \in R^{n_0}$ is a known vector, in the vicinity of which is an unknown vector x_0 of the initial condition [4].

In addition, in (4), we denote by $L_2(t_0, T)$ the set of vector-integrated vector functions, i.e.

$$L_2(t_0, T) = \left\{ f(t) \in R^n :\ \int_{t_0}^{T} f^T(t) f(t) dt = \int_{t_0}^{T} \| f(t) \|^2 dt < \infty \right\}.$$

The purpose of the article. The task is to find the optimal control $u(t)$ in the form of feedback on the output, which minimizes the functional (3) with the most adverse perturbations $\xi = (w(t), v(t), x_0)$ acting on the object and in the observation channel.

Formally, this task can be represented as a minimal maximal control problem

$$\inf_u \sup_{\xi \in \Omega_\xi} I(u), \tag{6}$$

where $I(u)$ — the functional of the form (3), Ω_ξ — set of permissible uncertainties (4).

It is convenient to solve this task in the task of H^∞-control. To do this, we first transform the quality criterion (3) accordingly, and then consider the feasibility of some assumptions that allow us to solve the problem.

Presenting main material. It is known [5] that any symmetric nonnegative matrix can be factored, that is, represent in the form $G = G^{1/2} \cdot G^{1/2}$ where the symmetric matrix $G^{1/2}$ can be found by means of the Choletsky procedure or through eigenvalues and matrix vectors G. So let us imagine the weight matrices $G_x(t)$, $G_u(t)$, G_f of functional (3) in the form

$$G_x(t) = G_x^{1/2}(t) \cdot G_x^{1/2}(t),\ G_u(t) = G_u^{1/2}(t) \cdot G_u^{1/2}(t),\ G_f = G_f^{1/2} \cdot G_f^{1/2}. \qquad (7)$$

Then criterion (3) can be transformed as follows

$$I(u) = \int_{t_0}^{T} \left(x^T(t) G_x^{1/2}(t) G_x^{1/2}(t) x(t) + u^T(t) G_u^{1/2}(t) G_u^{1/2}(t) u(t) \right) dt + x^T(T) G_f^{1/2} G_f^{1/2} x(T) =$$

$$= \int_{t_0}^{T} z^T(t) z(t) dt + z^T(T) z(T) = \| z \|^2,$$

where checked

$$z(t) = \begin{pmatrix} G_x^{1/2}(t) x(t) \\ G_u^{1/2}(t) u(t) \end{pmatrix},\ z(T) = G_f^{1/2} x(T),$$

and the norm $\| z \|^2$ is defined for the vector $z = \begin{pmatrix} z(t) \\ z(T) \end{pmatrix}$.

The vector $z(t)$ can be represented and so

$$z(t) = \begin{pmatrix} G_x^{1/2}(t) \\ 0 \end{pmatrix} x(t) + \begin{pmatrix} 0 \\ G_u^{1/2}(t) \end{pmatrix} u(t) \qquad (8)$$

and interpret it together with the vector $z(T)$ as adjustable quantities.

Since system (1) is linear, there is a linear operator that maps (converts) the vector of external input influences acting on the system and the observation channel into a vector of regulated quantities z, that is $z = R(\xi)$. In view of this, let us transform expression (6) $\sup\limits_{\xi \in \Omega_\xi} I(u)$, which describes the most negative effect of disturbances on a control object in the sense of increasing the value of criterion (3)

$$\sup_{\xi \in \Omega_\xi} I(u) = \sup_{\xi \in \Omega_\xi} \| z \|^2 = \sup_{\|\xi\|^2 \le 1} \| R(\xi) \|^2 = \sup_{\|\xi\| \le 1} \| R(\xi) \|^2 = \sup_{\xi,\ \xi \ne 0} \frac{\| R(\xi) \|^2}{\| \xi \|^2}. \qquad (9)$$

If we denote the last expression (9) by γ^2, that is

$$\sup_{\xi,\ \xi \ne 0} \frac{\| R(\xi) \|^2}{\| \xi \|^2} = \gamma^2$$

then we get the obvious inequality

$$\frac{\| R(\xi) \|^2}{\| \xi \|^2} < \gamma^2 \quad \forall \xi \ (\| \xi \| \ne 0) \quad \text{or} \quad \frac{\| z \|^2}{\| \xi \|^2} < \gamma^2 \quad \forall \xi \ (\| \xi \| \ne 0), \qquad (10)$$

the left part of which can be interpreted as the relative energy of the output signal z to the input ξ, and the right part γ^2 — as the limiting (maximum) value of this energy.

Inequality (10) underlies the theory of H^∞-control [6]. The task of finding a control $u(t)$ that ensures that inequality (10) is satisfied at a given value γ^2 is known as the problem of extinguishing external perturbations [7]. Thus, the output of the minimax control problem will now be reduced to the H^∞ - control problem.

Further, in accordance with the general search method H^{∞}-control, we introduce functional

$$J(u,\xi)=\| z\|^2-\gamma^2\|\xi\|^2=I(u)-\gamma^2\|\xi\|^2, \tag{11}$$

for which we find a point that satisfies the condition

$$J(u^*,\xi^*)=\min_u\max_\xi J(u,\xi). \tag{12}$$

A vector u^* is the desired control in which inequality (10) is given at a given value γ^2, and ξ^* is the most adverse perturbation.

If control u is seen as a designer trying to minimize losses and perturbation ξ as a nature that resists the designer and tries to maximize his losses, then we have a dynamic game problem. It belongs to the class of zero-sum differential games and the price of the game described by the functional $J(u,\xi)$. If (u^*,ξ^*) is the saddle point of the game task, that is, the point that satisfies the condition

$$J(u^*,\xi)\le J(u^*,\xi^*)\le J(u,\xi^*),$$

then relation (12) determines the upper value of the price of the game.

To solve problem (12), we transform pre-function (11). Substituting expressions (3) and (5) into (11), and given that $\xi=(w,v,x_0)$, we obtain

$$J(u,v,w,x_0)=x^T(T)G_f x(T)-\gamma^2\left(x_0-\hat{x}_0\right)^T R_0\left(x_0-\hat{x}_0\right)+$$
$$+\int_{t_0}^{T}\left[x^T(t)G_x(t)x(t)+u^T(t)G_u(t)u(t)-\gamma^2\left(w^T(t)R_w(t)w(t)+v^T(t)R_v(t)v(t)\right)\right]dt. \tag{13}$$

Then problem (12) is transformed to the next expression

$$J(u^*,v^*,w^*,x_0^*)=\min_u\max_v\max_w\max_{x_0}J(u,v,w,x_0). \tag{14}$$

We solve the problem (14) in two stages:

a) first, solve the intermediate problem

$$J_0(u,v)=J(u,v,w^*,x_0^*)=\max_w\max_{x_0}J(u,v,w,x_0) \tag{15}$$

at fixed vectors u and v;

b) then find the final optimal control by solving the following dynamic game problem

$$J(u^*,v^*,w^*,x_0^*)=\min_u\max_v J(u,v,w^*,x_0^*). \tag{16}$$

Once again, we transform the functional (13). To do this, expressing the vector of interference $v(t)$ from the equation of observations (2) and substituting it in (13), we obtain

$$J(u,v,w,x_0)=\| x(T)\|^2_{G_f}-\gamma^2\| x_0-\hat{x}_0\|^2_{R_0}+$$
$$+\int_{t_0}^{T}\left[\| x(t)\|^2_{G_x(t)}+\| u(t)\|^2_{G_u(t)}-\gamma^2\left(\| w(t)\|^2_{R_w(t)}+\| y(t)-C(t)x(t)\|^2_{R(t)}\right)\right]dt, \tag{17}$$

where

$$R(t)=(F_v^{-1}(t))^T R_v(t)F_v^{-1}(t). \tag{17.1}$$

Because the optimization parameters of problem (15) are heterogeneous, that is, $w(t)$ — a vector function and x_0 — a vector of constants, then to solve this optimization problem we use the methods of variational calculus, namely, we use the necessary and sufficient conditions of the extremum of the functional (15) in which both the first and second variations. Let us also dwell on the method of calculating them.

Variation of the quality criterion of the optimal control problem.

Suppose a controlled object is described by a system of differential equations [8]

$$\begin{cases} \dot{x}(t) = \dfrac{dx(t)}{dt} = f(x,u,t), \qquad t_0 < t \le T, \\ x(t_0) = h(x_0), \end{cases} \tag{18}$$

where $x = x(t)$ — is the state vector, $u = u(t)$ is the control vector, $f(x,u,t)$, $h(x_0)$ — are the known analytical vector functions of the corresponding dimensions.

Consider the criterion of quality of functioning of the object in the following form

$$J(u,x_0) = \varphi(x_0, x(T)) + \int_{t_0}^{T} g(x,u,t)dt, \tag{19}$$

where $\varphi(x_0, x(T))$, $g(x,u,t)$ — given scalar functions.

We consider the problem of finding optimal control $u(t)$ and initial state x_0 that maximize the criterion (19) in the optimization problem

$$J(u,x_0) \to \max_{u,x_0}. \tag{20}$$

To solve it, we use the Lagrange multiplier method, according to which we introduce as a criterion auxiliary functional

$$I(u,x_0,\lambda) = \varphi(x_0, x(T)) + \int_{t_0}^{T} g(x,u,t)dt + \int_{t_0}^{T} \lambda^T(t)\big(f(x,u,t) - \dot{x}\big)dt, \tag{21}$$

where $\lambda(t)$ is the vector column of Lagrange multipliers.

For convenience, we also introduce an auxiliary scalar function $H(x,u,\lambda,t)$ called the Hamiltonian

$$H(x,u,\lambda,t) = g(x,u,t) + \lambda^T(t) f(x,u,t). \tag{22}$$

And taking into account the notation (22), we transform the functional (21)

$$\begin{aligned} I(u,x_0,\lambda) = \varphi(x_0, x(T)) - \lambda^T(T)x(T) + \lambda^T(t_0)h(x_0) + \\ + \int_{t_0}^{T} \Big(H(x,u,\lambda,t) + \dot{\lambda}^T(t)x(t)\Big)dt \end{aligned} \tag{23}$$

To find the maximum value of the functional $I(u,x_0,\lambda)$, the necessary condition of the extremum of the first kind is used, namely, in order for the functional $I(u,x_0,\lambda)$ to reach its extreme value, it is necessary for its variation $\delta I(u,x_0,\lambda) = 0$ to be equal to zero for all variations $\delta u(t)$ and δx_0, and they do not rotate simultaneously to zero [9].

We find the first variation of criterion (23) corresponding to the variations of the control vector $u(t)$ and the initial condition x_0 (for fixed t_0 and T)

$$\delta I(u,x_0,\lambda)=\left(\frac{\partial\varphi(x_0,x(T))}{\partial x_0}+\frac{\partial h^T(x_0)}{\partial x_0}\lambda(t_0)\right)^T\delta x_0+ \\ +\left(\frac{\partial\varphi(x_0,x(T))}{\partial x(T)}-\lambda(T)\right)^T\delta x(T)+ \\ +\int_{t_0}^{T}\left[\left(\frac{\partial H(x,u,\lambda,t)}{\partial x}+\dot{\lambda}(t)\right)^T\delta x(t)+\left(\frac{\partial H(x,u,\lambda,t)}{\partial u}\right)^T\delta u(t)\right]dt \quad , \tag{24}$$

where $\delta x(t)$ — a variation of the state $x(t)$ corresponding to the variations of the initial state δx_0 and control $\delta u(t)$.

Note that upon receipt of variation (24), the following formulas for calculating the first variations were used

$$\delta\varphi(x_0,x(T))=\left(\frac{\partial\varphi(x_0,x(T))}{\partial x_0}\right)^T\delta x_0+\left(\frac{\partial\varphi(x_0,x(T))}{\partial x(T)}\right)^T\delta x(T),$$

$$\delta H(x,u,\lambda,t)=\left(\frac{\partial H(x,u,\lambda,t)}{\partial x}\right)^T\delta x(t)+\left(\frac{\partial H(x,u,\lambda,t)}{\partial u}\right)^T\delta u(t),$$

$$\delta\left(h^T(x_0)\lambda(t_0)\right)=\left(\frac{\partial h^T(x_0)}{\partial x_0}\lambda(t_0)\right)^T\delta x_0=\lambda^T(t_0)\left(\frac{\partial h^T(x_0)}{\partial x_0}\right)^T\delta x_0.$$

Given the arbitrariness of variations δx_0 and $\delta u(t)$ (which do not rotate at the same time simultaneously), the necessary condition of the extremum ($\delta I(u,x_0,\lambda)=0$) of the functional $I(u,x_0,\lambda)$ implies

$$\frac{\partial H(x,u,\lambda,t)}{\partial x}+\dot{\lambda}(t)=0,\quad \frac{\partial H(x,u,\lambda,t)}{\partial u}=0, \tag{25}$$

$$\frac{\partial\varphi(x_0,x(T))}{\partial x_0}+\frac{\partial h^T(x_0)}{\partial x_0}\lambda(t_0)=0,\quad \frac{\partial\varphi(x_0,x(T))}{\partial x(T)}-\lambda(T)=0, \tag{26}$$

And it follows from (25) that

$$\dot{\lambda}(t)=-\frac{\partial H(x,u,\lambda,t)}{\partial x}=-\frac{\partial g(x,u,t)}{\partial x}-\frac{\partial f^T(x,u,t)}{\partial x}\lambda(t), \tag{27}$$

$$\frac{\partial H(x,u,\lambda,t)}{\partial u}=\frac{\partial g(x,u,t)}{\partial u}+\frac{\partial f^T(x,u,t)}{\partial u}\lambda(t)=0. \tag{28}$$

Thus, the initial state x_0 and control vector $u(t)$ are necessary and certainly determined from the equations

$$\frac{\partial\varphi(x_0,x(T))}{\partial x_0}+\frac{\partial h^T(x_0)}{\partial x_0}\lambda(t_0)=0, \tag{29}$$

$$\frac{\partial H(x,u,\lambda,t)}{\partial u}=\frac{\partial g(x,u,t)}{\partial u}+\frac{\partial f^T(x,u,t)}{\partial u}\lambda(t)=0\,, \tag{30}$$

where vectors $x(t)$ and $\lambda(t)$ are exactly the solutions of the following system of conjugate equations (two-point boundary value problem)

$$\begin{cases}\dot{x}(t)=f(x,u,t), \quad t_0<t\le T,\\ x(t_0)=h(x_0),\end{cases} \tag{31}$$

$$\begin{cases}\dot{\lambda}(t)=-\dfrac{\partial f^T(x,u,t)}{\partial x}\lambda(t)-\dfrac{\partial g(x,u,t)}{\partial x},\\ \lambda(T)=\dfrac{\partial \varphi(x_0,x(T))}{\partial x(T)}.\end{cases} \tag{32}$$

To solve this optimization problem, a necessary condition of the extremum of the second kind can also be used [10]: in order for the functional (21) to reach the maximum value, it is necessary that the second variation $\delta^2 I(u,x_0,\lambda)$ is non-positive, that is $\delta^2 I(u,x_0,\lambda)\le 0$, for all non-zero simultaneously variations of arguments $\delta u(t)$ and δx_0.

Note that the second variation $\delta^2 I(u,x_0,\lambda)$ is determined by the following square form

$$\begin{gathered}\delta^2 I(u,x_0,\lambda)=\delta x_0^T\left[\frac{\partial}{\partial x_0}\left(\frac{\partial\varphi}{\partial x_0}+\frac{\partial h^T(x_0)}{\partial x_0}\lambda(t_0)\right)^T\right]\delta x_0+\\ +\delta x^T(T)\left[\frac{\partial}{\partial x(T)}\left(\frac{\partial\varphi}{\partial x(T)}\right)^T\right]\delta x(T)+\\ +\int_{t_0}^{T}\left\{\delta x^T(t)\left[\frac{\partial}{\partial x}\left(\frac{\partial H}{\partial x}\right)^T\right]\delta x(t)+\delta x^T(t)\left[\frac{\partial}{\partial x}\left(\frac{\partial H}{\partial u}\right)^T\right]\delta u(t)+\right.\\ \left.+\delta u^T(t)\left[\frac{\partial}{\partial u}\left(\frac{\partial H}{\partial x}\right)^T\right]\delta x(t)+\delta u^T(t)\left[\frac{\partial}{\partial u}\left(\frac{\partial H}{\partial u}\right)^T\right]\delta u(t)\right\}dt\end{gathered}\,, \tag{33}$$

where $H=H(x,u,\lambda,t)$ is the Hamiltonian function of the form (22).

A sufficient condition of the extremum of a functional $I(u,x_0,\lambda)$ is determined by inequality $\delta^2 I(u,x_0,\lambda)<0$.

The following vector differentiation formulas were used to find variations [11]:

a) if $f(x,u)$ is the scalar function of vector arguments and $\dfrac{\partial}{\partial x}$ — gradient operator of the form (vector column)

$$\frac{\partial}{\partial x} = \begin{pmatrix} \frac{\partial}{\partial x_1} \\ \frac{\partial}{\partial x_2} \\ \dots \\ \frac{\partial}{\partial x_n} \end{pmatrix},$$

then the gradient of the function $f(x,u)$ is equal to

$$f_x(x,u) = \frac{\partial f(x,u)}{\partial x} = \frac{\partial}{\partial x} f(x,u) = \begin{pmatrix} \frac{\partial}{\partial x_1} \\ \frac{\partial}{\partial x_2} \\ \dots \\ \frac{\partial}{\partial x_n} \end{pmatrix} f(x,u) = \begin{pmatrix} \frac{\partial f(x,u)}{\partial x_1} \\ \frac{\partial f(x,u)}{\partial x_2} \\ \dots \\ \frac{\partial f(x,u)}{\partial x_n} \end{pmatrix};$$

b) if $f(x,u)$ is the vector function of the vector arguments of the form

$$f(x,u) = \begin{pmatrix} f_1(x,u) \\ f_2(x,u) \\ \dots \\ f_n(x,u) \end{pmatrix},$$

then her Jacobian is equal

$$f_x(x,u) = \left(\frac{\partial f^T(x,u)}{\partial x} \right)^T = \begin{pmatrix} \frac{\partial f_1(x,u)}{\partial x_1} & \frac{\partial f_1(x,u)}{\partial x_2} & \dots & \frac{\partial f_1(x,u)}{\partial x_n} \\ \frac{\partial f_2(x,u)}{\partial x_1} & \frac{\partial f_2(x,u)}{\partial x_2} & \dots & \frac{\partial f_2(x,u)}{\partial x_n} \\ \dots & \dots & \dots & \dots \\ \frac{\partial f_n(x,u)}{\partial x_1} & \frac{\partial f_n(x,u)}{\partial x_2} & \dots & \frac{\partial f_n(x,u)}{\partial x_n} \end{pmatrix};$$

c) if a is a vector of constants, then

$$\frac{\partial}{\partial x}\left(a^T f(x,u)\right) = \frac{\partial}{\partial x}\left(f^T(x,u)a\right) = \frac{\partial f^T(x,u)}{\partial x} a.$$

Recall also the second necessary condition of the extremum of the functional using the second variation $\delta^2 I(u,x_0,\lambda)$ (and which will be used in the future). In order for the functional $I(u,x_0,\lambda)$ to reach its extreme value, it is necessary that the second variation $\delta^2 I(u,x_0,\lambda)$ be $\delta^2 I(u,x_0,\lambda)\le 0$ for all variations δx_0 and $\delta u(t)$ that do not simultaneously rotate to zero.

When looking for the second variation of the functional, we will further use the following formulas [12]:

a) if $f(x,u)$ — scalar function, then

$$\delta f(x,u)=\left(\frac{\partial f(x,u)}{\partial x}\right)^T\delta x+\left(\frac{\partial f(x,u)}{\partial u}\right)^T\delta u\,;$$

b) if $f(x,u)$ — a vector function, and a — a vector of constants then

$$\delta\left(f^T(x,u)a\right)=a^T\left(\frac{\partial f^T(x,u)}{\partial x}\right)^T\delta x+a^T\left(\frac{\partial f^T(x,u)}{\partial u}\right)^T\delta u\,;$$

c) if $f(x,u)$ the scalar function of a vector argument $f(x)$, then

$$\delta^2 f(x,u)=\delta x^T\left[\frac{\partial}{\partial x}\left(\frac{\partial f(x,u)}{\partial x}\right)^T\right]\delta x+\delta x^T\left[\frac{\partial}{\partial x}\left(\frac{\partial f(x,u)}{\partial u}\right)^T\right]\delta u+$$

$$+\delta u^T\left[\frac{\partial}{\partial u}\left(\frac{\partial f(x,u)}{\partial x}\right)^T\right]\delta x+\delta u^T\left[\frac{\partial}{\partial u}\left(\frac{\partial f(x,u)}{\partial u}\right)^T\right]\delta u=$$

$$=\begin{pmatrix}\delta x\\ \delta u\end{pmatrix}^T\left(\begin{array}{c|c}\dfrac{\partial}{\partial x}\left(\dfrac{\partial f(x,u)}{\partial x}\right)^T & \dfrac{\partial}{\partial x}\left(\dfrac{\partial f(x,u)}{\partial u}\right)^T\\ \hline \dfrac{\partial}{\partial u}\left(\dfrac{\partial f(x,u)}{\partial x}\right)^T & \dfrac{\partial}{\partial u}\left(\dfrac{\partial f(x,u)}{\partial x}\right)^T\end{array}\right)\begin{pmatrix}\delta x\\ \delta u\end{pmatrix}.$$

In particular, for a scalar function of a vector argument, the second variation is determined by the following square form

$$\delta^2 f(x)=\delta x^T\left[\frac{\partial}{\partial x}\left(\frac{\partial f(x)}{\partial x}\right)^T\right]\delta x\,.$$

Solution of the auxiliary optimization problem (continued).

Let us now return to the problem of finding the maximum value of the functional (17). To solve this problem, we use the Lagrange multiplier method, according to which we introduce functional

$$L(w,x_0,\lambda)=J(u,v,w,x_0)+$$
$$+\int_{t_0}^{T}\lambda^T(t)\left(A(t)x(t)+B(t)u(t)+F_w(t)w(t)-\dot{x}(t)\right)dt\,, \tag{34}$$

where $\lambda(t)$ is the vector of Lagrange multipliers, and the functional $J(u,v,w,x_0)$ is determined by (17).

We next use the necessary first-order extremum condition for the functional (34) ($\delta L(u,w,x_0)=0$) and the results of the previous paragraph. Given that for our optimization problem, the functions $f(x,w,t)$, $h(x_0)$, $\varphi(x_0,x(T))$, $g(x,w,t)$ are equal

$$f(x,w,t)=A(t)x(t)+B(t)u(t)+F_w(t)w(t),\ h(x_0)=F_0x_0,$$
$$\varphi(x_0,x(T))=x^T(T)G_fx(T)-\gamma^2\left(x_0-\hat{x}_0\right)^T R_0\left(x_0-\hat{x}_0\right),$$
$$g(x,w,t)=x^T(t)G_x(t)x(t)+u^T(t)G_u(t)u(t)-\gamma^2w^T(t)R_w(t)w(t)-$$
$$-\gamma^2\left(y(t)-C(t)x(t)\right)^T R(t)\left(y(t)-C(t)x(t)\right),$$
$$R(t)=(F_v^{-1}(t))^T R_v(t)F_v^{-1}(t),$$

the solution of the problem $\max\limits_{x_0,w} L(u,w,x_0)$ due to the necessary condition of the extremes of the first order is from the equations

$$\frac{\partial\varphi(x_0,x(T))}{\partial x_0}+\frac{\partial h^T(x_0)}{\partial x_0}\lambda(t_0)=0,$$
$$\frac{\partial g(x,w,t)}{\partial w}+\frac{\partial f^T(x,w,t)}{\partial w}\lambda(t)=0,$$

that are converted to appearance

$$F_0^T\lambda(t_0)-2\gamma^2R_0(x_0-\hat{x}_0)=0,$$
$$-2\gamma^2R_w(t)w(t)+F_w^T(t)\lambda(t)=0.$$

From here, we find the relations that satisfy the vectors $w(t)$ and x_0

$$x_0=\frac{1}{2}\gamma^{-2}R_0^{-1}F_0^T\lambda(t_0)+\hat{x}_0, \tag{35}$$
$$w(t)=\frac{1}{2}\gamma^{-2}R_w^{-1}(t)F_w^T(t)\lambda(t), \tag{36}$$

where the vector function $\lambda(t)$ is the solution of the following system of conjugate equations

$$\begin{cases}\dfrac{d\lambda(t)}{dt}=-\dfrac{\partial f^T(x,w,t)}{\partial x}\lambda(t)-\dfrac{\partial g(x,w,t)}{\partial x},\\ \lambda(T)=\dfrac{\partial\varphi(x_0,x(T))}{\partial x(T)}.\end{cases}$$

This system, after the transformations, is as follows

$$\begin{cases}\dfrac{d\lambda(t)}{dt}=-A^T(t)\lambda(t)-2G_x(t)x(t)-2\gamma^2C^T(t)R(t)\left(y(t)-C(t)x(t)\right),\\ \lambda(T)=2G_fx(T),\end{cases} \tag{37}$$

where $x(t)$ is the solution of the system

$$\begin{cases}\dfrac{dx(t)}{dt}=A(t)x(t)+B(t)u(t)+F_w(t)w(t), \quad t_0<t<T,\\ x(t_0)=F_0x_0,\end{cases} \tag{38}$$

After substituting (35) and (36) into equation (38), we obtain a two-point boundary value problem

$$\frac{d}{dt}\begin{pmatrix} x(t) \\ \lambda(t) \end{pmatrix} = \begin{pmatrix} A(t) & \frac{1}{2}\gamma^{-2}F_w(t)R_w^{-1}(t)F_w^T(t) \\ -2G_x(t)+2\gamma^2C^T(t)R(t)C(t) & -A^T(t) \end{pmatrix}\begin{pmatrix} x(t) \\ \lambda(t) \end{pmatrix} + \\ +\begin{pmatrix} B(t)u(t) \\ -2\gamma^2C^T(t)R(t)y(t) \end{pmatrix}, \quad (39)$$

with boundary conditions

$$x(t_0)=\frac{1}{2}\gamma^{-2}F_0R_0^{-1}F_0^T\lambda(t_0)+F_0\hat{x}_0,\ \ \lambda(T)=2G_fx(T). \quad (40)$$

The following formulas were used to obtain equations (35), (36) and system (37)

$$\frac{\partial\varphi(x_0,x(T))}{\partial x_0}=-2\gamma^2R_0(x_0-\hat{x}_0),\ \ \frac{\partial\varphi(x_0,x(T))}{\partial x(T)}=2G_fx(T),$$

$$\frac{\partial g(x,w,t)}{\partial x}=2G_xx-2\gamma^2\left(C^TRCx-C^TRy\right),\ \ \frac{\partial g(x,w,t)}{\partial w}=-2\gamma^2R_ww,$$

$$\frac{\partial}{\partial x}f^T(x,w,t)\lambda=A^T\lambda,\ \ \frac{\partial}{\partial w}f^T(x,w,t)\lambda=F_w^T\lambda,\ \ \frac{\partial}{\partial x_0}h^T(x_0)\lambda(t_0)=F_0^T\lambda(t_0).$$

Since boundary-value problem (39) is linear, we can assume that the solution can be represented as

$$x(t)=\hat{x}(t)+\frac{1}{2}\gamma^{-2}P(t)\lambda(t), \quad (41)$$

where $\hat{x}(t)$ and $P(t)$ are the unknown vector and matrix to be determined.

Differentiating (41) and using the conjugate system (37), we obtain

$$\frac{1}{2}\gamma^{-2}\left[\dot{P}(t)-PA^T-\gamma^{-2}PG_xP+PC^TRCP-AP-F_wR_w^{-1}F_w^T\right]\lambda+ \\ +\left[-\gamma^{-2}PG_x\hat{x}+PC^TRC\hat{x}-PC^TRy+\dot{\hat{x}}-A\hat{x}-Bu\right]=0 \quad (42)$$

If put

$$\dot{P}=AP+PA^T-P\left(C^TRC-\gamma^{-2}G_x\right)P+F_wR_w^{-1}F_w^T, \quad (43)$$

$$\dot{\hat{x}}=A\hat{x}+Bu+\gamma^{-2}PG_x\hat{x}+PC^TR\left(y-C\hat{x}\right), \quad (44)$$

then (42) becomes an identity.

We now find the initial conditions for equations (43) and (44). To do this, substitute in (41) $t=t_0$ and obtain an expression for the initial conditions

$$x(t_0)=\hat{x}(t_0)+\frac{1}{2}\gamma^{-2}P(t_0)\lambda(t_0).$$

Given (35) and the initial conditions (37), the latter relation is transformed to the form

$$F_0\hat{x}_0+\frac{1}{2}\gamma^{-2}F_0R_0^{-1}F_0^T\lambda(t_0)=\hat{x}(t_0)+\frac{1}{2}\gamma^{-2}P(t_0)\lambda(t_0),$$

where will we get it from

$$P(t_0) = F_0 R_0^{-1} F_0^T,\ \hat{x}(t_0) = F_0 \hat{x}_0.$$

Thus, to determine $P(t)$ and $\hat{x}(t)$ obtain the following equations

$$\begin{cases} \dot{P} = AP + PA^T - P\left(C^T RC - \gamma^{-2} G_x\right)P + F_w R_w^{-1} F_w^T, \\ P(t_0) = F_0 R_0^{-1} F_0^T, \end{cases} \tag{45}$$

$$\begin{cases} \dot{\hat{x}} = A\hat{x} + Bu + \gamma^{-2} PG_x \hat{x} + PC^T R\left(y - C\hat{x}\right), \\ \hat{x}(t_0) = F_0 \hat{x}_0. \end{cases} \tag{46}$$

Next, replace the variables

$$\mu(t) = \frac{1}{2}\gamma^{-2}\lambda(t)$$

and represent the optimal vectors x_0 and $w(t)$ in the form

$$x_0 = R_0^{-1} F_0^T \mu(t_0) + \hat{x}_0, \tag{47}$$

$$w(t) = R_w^{-1}(t) F_w^T(t) \mu(t), \tag{48}$$

where $\mu(t)$ is the solution of the following system of equations

$$\begin{cases} \dfrac{d\mu(t)}{dt} = -A^T(t)\mu(t) - \gamma^{-2} G_x(t) x(t) - C^T(t) R(t)\left(y(t) - C(t) x(t)\right), \\ \mu(T) = \gamma^{-2} G_f x(T). \end{cases} \tag{49}$$

Let us now transform system (49) to a form that depends not on state $x(t)$ but on $\hat{x}(t)$. Taking into account that

$$x(t) = \hat{x}(t) + P(t)\mu(t), \tag{50}$$

we transform the initial conditions for system (49) as follows

$$\mu(T) = \gamma^{-2} G_f\left(\hat{x}(T) + P(T)\mu(T)\right),$$

where from

$$\mu(T) = \left(\gamma^2 E - G_f P(T)\right)^{-1} G_f \hat{x}(T).$$

Then the system (49) itself takes the form

$$\frac{d\mu(t)}{dt} = \left(-A^T(t) - \gamma^{-2} G_x(t) P(t) + C^T(t) R(t) C(t) P(t)\right)\mu(t) - \gamma^{-2} G_x(t)\hat{x}(t) - \\ - C^T(t) R(t)\left(y(t) - C(t)\hat{x}(t)\right).$$

As a result, we get

$$\begin{cases} \dfrac{d\mu(t)}{dt} = \left(-A^T(t) - \gamma^{-2} G_x(t) P(t) + C^T(t) R(t) C(t) P(t)\right)\mu(t) - \\ \qquad - \gamma^{-2} G_x(t)\hat{x}(t) - C^T(t) R(t)\left(y(t) - C(t)\hat{x}(t)\right), \\ \mu(T) = \left(\gamma^2 E - G_f P(T)\right)^{-1} G_f \hat{x}(T). \end{cases} \tag{51}$$

Note that system (51) will be used in further transformations.

We now use the necessary conditions of optimality of the second kind. To do this, we find the second variation of the functional (34)

$$\delta^2 L(u,w,x_0) = -2\gamma^2 \delta x_0^T R_0 \delta x_0 + 2\delta x^T(T) G_f \delta x(T) +$$
$$+2\int_{t_0}^{T} \left\{ \delta x^T(t)\left(G_x(t) - \gamma^2 C^T(t) R(t) C(t)\right)\delta x(t) - \gamma^2 \delta w^T(t) R_w(t) \delta w(t) \right\} dt, \tag{52}$$

where $\delta x(t)$ is the solution of the equation in variations

$$\begin{cases} \delta\dot{x}(t) = A(t)\delta x(t) + F_w(t)\delta w(t), \\ \delta x(t_0) = F_0 \delta x_0. \end{cases} \tag{53}$$

Let us transform the relation (52). Add to the expression $\delta^2 L(u,w,x_0)$ that is equal to zero

$$-2\gamma^2 \left[\delta x^T(t) P^{-1}(t) \delta x(t) \Big|_{t_0}^{T} - \int_{t_0}^{T} \frac{d}{dt}\left(\delta x^T(t) P^{-1}(t) \delta x(t) \right) dt \right] = 0 .$$

Then we get

$$\begin{aligned} &\delta^2 L(u,w,x_0) = -2\gamma^2 \delta x_0^T R_0 \delta x_0 + 2\delta x^T(T) G_f \delta x(T) + \\ &+2\int_{t_0}^{T} \left\{ \delta x^T(t)\left(G_x(t) - \gamma^2 C^T(t) R(t) C(t)\right)\delta x(t) - \gamma^2 \delta w^T(t) R_w(t) \delta w(t) \right\} dt - \\ &-2\gamma^2 \delta x^T(T) P^{-1}(T) \delta x(T) + 2\gamma^2 \delta x^T(t_0) P^{-1}(t_0) \delta x(t_0) + 2\gamma^2 \int_{t_0}^{T} \Big\{ \Big(\delta x^T(t) A^T(t) + \\ &+\delta w^T(t) F_w^T(t) \Big) P^{-1}(t) \delta x(t) + \delta x^T(t) P^{-1}(t) \left(A(t)\delta x(t) + F_w(t) \delta w(t) \right) - \\ &-\delta x^T(t) P^{-1}(t) \left(AP + PA^T - P\left(C^T R C - \gamma^{-2} G_x \right) P + F_w R_w^{-1} F_w^T \right) P^{-1}(t) \delta x(t) \Big\} dt = \\ &= 2\gamma^2 \delta x_0^T \left[F_0^T \left(F_0 R_0^{-1} F_0^T \right)^{-1} F_0 - R_0 \right] \delta x_0 + 2\delta x^T(T) \left[G_f - \gamma^2 P^{-1}(T) \right] \delta x(T) - \\ &-2\gamma^2 \int_{t_0}^{T} \left\{ \begin{array}{l} \left(\delta w(t) - R_w^{-1}(t) F_w^T(t) P^{-1}(t) \delta x(t) \right)^T \times \\ \times R_w(t) \left(\delta w(t) - R_w^{-1}(t) F_w^T(t) P^{-1}(t) \delta x(t) \right) \end{array} \right\} dt. \end{aligned} \tag{54}$$

Note that the matrix equation was used in these transformations

$$\frac{dP^{-1}(t)}{dt} = -P^{-1}(t) \frac{dP(t)}{dt} P^{-1}(t) .$$

From the last equation it follows

$$\left(R_0 - F_0^T \left(F_0 R_0^{-1} F_0^T \right)^{-1} F_0 \right) \delta x_0 = 0 . \tag{55}$$

And from (48) and (50) it turns out

$$w(t) = R_w^{-1}(t) F_w^T(t) P^{-1}(t) \left(x(t) - \hat{x}(t) \right),$$

where from

$$\delta w(t) = R_w^{-1}(t) F_w^T(t) P^{-1}(t) \delta x(t) . \tag{56}$$

Given (55) and (56), the second variation of the functional will take the form

$$\delta^2 L(u,w,x_0) = 2\delta x^T(T) \left[G_f - \gamma^2 P^{-1}(T) \right] \delta x(T) .$$

If the matrix $G_f - \gamma^2 P^{-1}(T)$ is negatively defined, that is

$$G_f - \gamma^2 P^{-1}(T) < 0, \quad (57)$$

then $\delta^2 L(u,w,x_0) < 0$, and hence, the quantities x_0 and $w(t)$ determined by relations (47), (48) satisfy not only the necessary but also sufficient conditions of the extrema of the functional $L(u,w,x_0)$, that is, the pair $(x_0, w(t))$ maximizes the functional $L(u,w,x_0)$, and hence the functional $J(u,v,w,x_0)$ under fixed control u and perturbation v.

We now find the value of the functional $J_0(u,v) = J(u,v,w,x_0)$ at the extremals $(x_0, w(t))$, that is, at optimal values of x_0 and $w(t)$

$$J_0(u,v) = x^T(T)G_f x(T) - \gamma^2 \mu^T(t_0) F_0 R_0^{-1} F_0^T \mu(t_0) + \\ + \int_{t_0}^{T} \{x^T(t)G_x(t)x(t) + u^T(t)G_u(t)u(t) - \\ -\gamma^2 \mu^T(t)F_w(t)R_w^{-1}(t)F_w^T(t)\mu(t) - \gamma^2 \left(y(t) - C(t)x(t)\right)^T R(t)\left(y(t) - C(t)x(t)\right)\} dt.$$

Adding to this expression a null value

$$-\gamma^2 \left[\mu^T(t)P(t)\mu(t)\Big|_{t_0}^{T} - \int_{t_0}^{T} \frac{d}{dt}\left(\mu^T(t)P(t)\mu(t)\right)dt \right] = 0,$$

obsessed

$$J_0(u,v) = x^T(T)G_f x(T) - \gamma^2 \mu^T(t_0) F_0 R_0^{-1} F_0^T \mu(t_0) + \\ + \int_{t_0}^{T} \begin{Bmatrix} x^T(t)G_x(t)x(t) + u^T(t)G_u(t)u(t) - \gamma^2 \mu^T(t)F_w(t)R_w^{-1}(t)F_w^T(t)\mu(t) - \\ -\gamma^2 \left(y(t) - C(t)x(t)\right)^T R(t)\left(y(t) - C(t)x(t)\right) \end{Bmatrix} dt - \\ -\gamma^2 \mu^T(T)P(t)\mu(T) + \gamma^2 \mu^T(t_0)P(t_0)\mu(t_0) + \\ +\gamma^2 \int_{t_0}^{T} \begin{Bmatrix} \left(-A^T(t)\mu(t) - \gamma^{-2}G_x(t)x(t) + C^T(t)R(t)\left(C(t)x(t) - y(t)\right)\right)^T P(t)\mu(t) + \\ +\mu^T(t)P(t)\left(-A^T(t)\mu(t) - \gamma^{-2}G_x(t)x(t) + C^T(t)R(t)\left(C(t)x(t) - y(t)\right)\right) + \\ +\mu^T(t)\left(A(t)P(t) + P(t)A^T(t) - P(t)\left(C^T(t)R(t)C(t) - \gamma^{-2}G_x(t)\right)P(t) + \right. \\ \left. + F_w(t)R_w^{-1}(t)F_w^T(t)\right)\mu(t) \end{Bmatrix} dt.$$

And after a series of further transformations we come to the expression

$$J_0(u,v) = x^T(T)G_f x(T) - \gamma^2 \mu^T(T)P(T)\mu(T) + \\ + \int_{t_0}^{T} \begin{Bmatrix} \hat{x}^T(t)G_x(t)\hat{x}(t) + u^T(t)G_u(t)u(t) - \\ -\gamma^2 \left(y(t) - C(t)\hat{x}(t)\right)^T R(t)\left(y(t) - C(t)\hat{x}(t)\right) \end{Bmatrix} dt. \quad (58)$$

Taking into account that

$$\mu(T) = \left(\gamma^2 E - G_f P(T)\right)^{-1} G_f \hat{x}(T),$$

we will find $x(T)$

$$x(T)=P(T)\mu(T)+\hat{x}(T)=\left[E+P(T)\left(\gamma^2E-G_fP(T)\right)^{-1}G_f\right]\hat{x}(T)=$$
$$=\left(E-\gamma^{-2}P(T)G_f\right)^{-1}\hat{x}(T)$$

Then you can get the same

$$x^T(T)G_fx(T)-\gamma^2\mu^T(T)P(T)\mu(T)=\hat{x}^T(T)\left[\left(E-\gamma^{-2}G_fP(T)\right)^{-1}G_f\times\right.$$
$$\left.\times\left(E-\gamma^{-2}P(T)G_f\right)^{-1}-\gamma^2G_f\left(\gamma^2E-P(T)G_f\right)^{-1}P(T)\left(\gamma^2E-G_fP(T)\right)^{-1}G_f\right]\hat{x}(T)=$$
$$=\hat{x}^T(T)\left(G_f^{-1}-\gamma^{-2}P(T)\right)^{-1}\left[G_f^{-1}\left(G_f^{-1}-\gamma^{-2}P(T)\right)^{-1}-\gamma^{-2}P(T)\left(G_f^{-1}-\gamma^{-2}P(T)\right)^{-1}\right]\times$$
$$\times\hat{x}(T)=\hat{x}^T(T)\left(G_f^{-1}-\gamma^{-2}P(T)\right)^{-1}\hat{x}(T).$$

Substituting the last expression into the functional (58) and making substitutions for the variables

$$\hat{v}(t)=y(t)-C(t)\hat{x}(t), \tag{59}$$

$$S_T=\left(G_f^{-1}-\gamma^{-2}P(T)\right)^{-1}, \tag{60}$$

finally convert the functionality to appearance

$$J_0(u,\hat{v})=\hat{x}^T(T)S_T\hat{x}(T)+$$
$$+\int_{t_0}^{T}\left\{\hat{x}^T(t)G_x(t)\hat{x}(t)+u^T(t)G_u(t)u(t)-\gamma^2\hat{v}^T(t)R(t)\hat{v}(t)\right\}dt. \tag{61}$$

Now we have to solve the minimax problem

$$\min_u\max_{\hat{v}}J_0(u,\hat{v}) \tag{62}$$

provided that $\hat{x}(t)$ satisfies the system

$$\begin{cases}\dot{\hat{x}}(t)=A_\gamma(t)\hat{x}(t)+B(t)u(t)+Q_v(t)\hat{v}(t),\\ \hat{x}(t_0)=F_0\hat{x}_0.\end{cases} \tag{63}$$

where

$$A_\gamma(t)=A(t)+\gamma^{-2}P(t)G_x(t),\ Q_v(t)=P(t)C^T(t)R(t). \tag{64}$$

To solve this problem, we use the results of the theory of linear-quadratic differential games.

Linear quadratic differential game problem of two people.

Consider the system

$$\begin{cases}\dfrac{dx(t)}{dt}=A(t)x(t)+B_1(t)u_1(t)+B_2(t)u_2(t),\\ x(t_0)=x_0,\end{cases}$$

with criterion

$$J(u_1,u_2)=x^T(T)Q_fx(T)+\int_{t_0}^{T}\left\{x^T(t)Q(t)x(t)+u_1^T(t)R_1(t)u_1(t)-u_2^T(t)R_2(t)u_2(t)\right\}dt,$$

where $x(t)$ — system status, $u_1(t)$, $u_2(t)$ — control functions.

System-forming matrices and criterion weight matrices are known.

You need to find controls $u_1(t)$ and $u_2(t)$ by condition

$$\min_{u_1}\max_{u_2} J(u_1,u_2).$$

The following result was obtained above: the optimal control strategy is determined by the functions of the species

$$u_1(t)=-R_1^{-1}(t)B_1^T(t)K(t)x(t),\ u_2(t)=R_2^{-1}(t)B_2^T(t)K(t)x(t),$$

where $K(t)$ is the solution of the matrix differential equation

$$\begin{cases}\dfrac{dK(t)}{dt}=-A^T(t)K(t)-K(t)A(t)+K(t)\big(S_1(t)-S_2(t)\big)K(t)-Q(t),\\ K(T)=Q_f,\end{cases}$$

in which

$$S_i(t)=B_i(t)R_i^{-1}(t)B_i^T(t),\ i=1,2.$$

The minimum value of the criterion is equal

$$\min_{u_1}\max_{u_2} J(u_1,u_2)=x_0^TK(t_0)x_0.$$

Applying this result to problem (61) — (64), we get its solution in the form

$$u(t)=-G_u^{-1}(t)B^T(t)S(t)\hat{x}(t), \tag{65}$$

$$\hat{v}(t)=\gamma^{-2}R^{-1}(t)Q_v^T(t)S(t)\hat{x}(t)=\gamma^{-2}C(t)P(t)S(t)\hat{x}(t), \tag{66}$$

$$J_0(u,\hat{v})=\hat{x}_0^TF_0^TS(t_0)F_0\hat{x}_0, \tag{67}$$

where $S(t)$ is the solution of the matrix differential equation of the form

$$\begin{cases}\dfrac{dS(t)}{dt}=-A_\gamma^T(t)S(t)-S(t)A_\gamma(t)-G_x(t)+\\ \qquad +S(t)\big(B(t)G_u^{-1}(t)B^T(t)-\gamma^{-2}Q_v(t)R^{-1}(t)Q_v^T(t)\big)S(t),\\ S(T)=S_T,\end{cases} \tag{68}$$

Substituting (65), (66) into (46) and (51), we obtain

$$\begin{cases}\dot{\hat{x}}(t)=\big(A(t)+\gamma^{-2}P(t)G_x(t)\big)\hat{x}(t)-\\ \qquad -\big(B(t)G_u^{-1}(t)B^T(t)-\gamma^{-2}P(t)C^T(t)R(t)C(t)P(t)\big)S(t)\hat{x}(t),\\ \hat{x}(t_0)=F_0\hat{x}_0.\end{cases} \tag{69}$$

$$\begin{cases}\dfrac{d\mu(t)}{dt}=-\big(A(t)+\gamma^{-2}P(t)G_x(t)\big)^T\mu(t)+C^T(t)R(t)C(t)P(t)\times\\ \qquad \times\big(\mu(t)-\gamma^{-2}S(t)\hat{x}(t)\big)-\gamma^{-2}G_x(t)\hat{x}(t),\\ \mu(T)=\big(\gamma^2E-G_fP(T)\big)^{-1}G_f\hat{x}(T)=\gamma^{-2}S_T\hat{x}(T).\end{cases} \tag{70}$$

We now denote the right-hand side of the second equation of system (70) as

$$\eta(t)=\gamma^{-2}S(t)\hat{x}(t)$$

and find an equation that satisfies $\eta(t)$. Using (68) and (69), we can show that

$$\begin{cases}\dfrac{d\eta(t)}{dt}=\gamma^{-2}\left(\dfrac{dS(t)}{dt}\hat{x}(t)+S(t)\dfrac{d\hat{x}(t)}{dt}\right)=\\ \qquad =-\left(A(t)+\gamma^{-2}P(t)G_x(t)\right)^T\eta(t)-\gamma^{-2}G_x(t)\hat{x}(t),\\ \eta(T)=\gamma^{-2}S_T\hat{x}(T)\end{cases}\tag{71}$$

Comparing equations (70) and (71), we conclude that $\mu(t)=\eta(t)$, $t\in[t_0,T]$ from where it follows

$$\mu(t)=\gamma^{-2}S(t)\hat{x}(t).\tag{72}$$

Now let us substitute $u(t)$ and $\mu(t)$ from the formulas (65) and (72) into equation (39). Then, given the relation (50), we have

$$\begin{cases}\dfrac{dx(t)}{dt}=A(t)x(t)+F_w(t)R_w^{-1}(t)F_w^T(t)\mu(t)-B(t)G_u^{-1}(t)B^T(t)S(t)\hat{x}(t)=\\ \qquad =A(t)x(t)+\left(\gamma^{-2}F_w(t)R_w^{-1}(t)F_w^T(t)-B(t)G_u^{-1}(t)B^T(t)\right)S(t)\hat{x}(t),\\ x(t_0)=\hat{x}(t_0)+\gamma^{-2}P(t_0)S(t_0)\hat{x}(t_0)=\left(E+\gamma^{-2}P(t_0)S(t_0)\right)\hat{x}(t_0).\end{cases}\tag{73}$$

Let us denote

$$h(t)=\left(E+\gamma^{-2}P(t)S(t)\right)\hat{x}(t).$$

And then, taking into account equations (45), (68), (69), it can be shown that

$$\begin{cases}\dfrac{dh(t)}{dt}=A(t)h(t)+\left(\gamma^{-2}F_w(t)R_w^{-1}(t)F_w^T(t)-B(t)G_u^{-1}(t)B^T(t)\right)S(t)\hat{x}(t),\\ h(t_0)=\left(E+\gamma^{-2}P(t_0)S(t_0)\right)\hat{x}(t_0).\end{cases}\tag{74}$$

Comparing systems (73) and (74), we conclude that, $x(t)=h(t)$, $t\in[t_0,T]$, and

$$x(t)=\left(E+\gamma^{-2}P(t)S(t)\right)\hat{x}(t).\tag{75}$$

We now find the optimum value for perturbation in the measurement channel. Given the observation equation (2) and the relation (59), (66), (75), we obtain

$$\begin{aligned}v(t)&=F_v^{-1}(t)\left(y(t)-C(t)x(t)\right)=F_v^{-1}(t)\left(\hat{v}(t)+C(t)\hat{x}(t)-C(t)x(t)\right)=\\ &=F_v^{-1}(t)\left(\gamma^{-2}C(t)P(t)S(t)\hat{x}(t)+C(t)\left(\hat{x}(t)-\left(E+\gamma^{-2}P(t)S(t)\right)\hat{x}(t)\right)\right)=0.\end{aligned}\tag{76}$$

Thus, optimum perturbation is $v(t)=0$.

Given the relation (72), we also transform the optimal perturbations x_0 and $w(t)$ determined by formulas (47) and (48)

$$x_0=R_0^{-1}F_0^T\mu(t_0)+\hat{x}_0=\gamma^{-2}R_0^{-1}F_0^TS(t_0)F_0\hat{x}_0+\hat{x}_0=\left(E+\gamma^{-2}R_0^{-1}F_0^TS(t_0)F_0\right)\hat{x}_0,\tag{77}$$

$$w(t)=R_w^{-1}(t)F_w^T(t)\mu(t)=\gamma^{-2}R_w^{-1}(t)F_w^T(t)S(t)\hat{x}(t).\tag{78}$$

The ratios that determine the optimal solution to the H^∞-control problem use the matrix, which is the solution of equation (68). This equation can be solved by knowing the matrix $P(t)$, which in turn is the solution of another matrix differential equation (45), that is, the matrix $S(t)$ is dependent on the matrix $P(t)$. In order to break this dependence, we introduce the following matrix

$$Q(t) = S(t)\left(E+\gamma^{-2}P(t)S(t)\right)^{-1} = \left(E+\gamma^{-2}S(t)P(t)\right)^{-1}S(t) = \left(S^{-1}(t)+\gamma^{-2}P(t)\right)^{-1}. \quad (79)$$

Given that $P(t)$ — the solution of equation (45) and $S(t)$ — satisfies equation (68), we can show that $Q(t)$ satisfies the matrix differential equation of the form

$$\begin{cases} \dfrac{dQ(t)}{dt} = -A^T(t)Q(t) - Q(t)A(t) - G_x(t) + \\ \qquad + Q(t)\left(B(t)G_u^{-1}(t)B^T(t) - \gamma^{-2}F_w(t)R_w^{-1}(t)F_w^T(t)\right)Q(t), \\ Q(T) = G_f, \end{cases} \quad (80)$$

It is also easy to show that the condition of boundedness of solutions (matrices) $P(t)$ and $S(t)$ and matrix differential equations (45) and (68)

$$G_f - \gamma^2 P^{-1}(T) < 0$$

is equivalent to this condition

$$E - \gamma^{-2}Q(t)P(t) > 0, \; t \in [t_0, T]. \quad (81)$$

Now we express from (79) the matrix $S(t)$ through $Q(t)$

$$\begin{aligned} S(t) &= \left(E-\gamma^{-2}Q(t)P(t)\right)^{-1}Q(t) = Q(t)\left(E-\gamma^{-2}P(t)Q(t)\right)^{-1} = \\ &= \left(Q^{-1}(t)-\gamma^{-2}P(t)\right)^{-1} \end{aligned}. \quad (82)$$

Then the solution to the problem H^∞-control can finally be represented as follows

$$u(t) = -G_u^{-1}(t)B^T(t)Q(t)\left(E-\gamma^{-2}P(t)Q(t)\right)^{-1}\hat{x}(t), \quad (83)$$

where the member $\hat{x}(t)$ is the solution to the next system

$$\begin{cases} \dfrac{d\hat{x}(t)}{dt} = A(t)\hat{x}(t) + B(t)u(t) + \gamma^{-2}P(t)G_x(t)\hat{x}(t) + \\ \qquad + P(t)C^T(t)R(t)\left(y(t) - C(t)\hat{x}(t)\right), \\ \hat{x}(t_0) = F_0\hat{x}_0. \end{cases} \quad (84)$$

And in turn, the member $R(t)$ is, respectively

$$R(t) = (F_v^{-1}(t))^T R_v(t) F_v^{-1}(t).$$

The matrices $P(t)$ and $Q(t)$ satisfy the following Riccati-type matrix equations

$$\begin{cases} \dfrac{dP(t)}{dt} = A(t)P(t) + P(t)A^T(t) + F_w(t)R_w^{-1}(t)F_w^T(t) - \\ \qquad - P(t)\left(C^T(t)R(t)C(t) - \gamma^{-2}G_x(t)\right)P(t), \\ P(t_0) = F_0R_0^{-1}F_0^T, \end{cases} \tag{85}$$

$$\begin{cases} \dfrac{dQ(t)}{dt} = -A^T(t)Q(t) - Q(t)A(t) - G_x(t) + \\ \qquad + Q(t)\left(B(t)G_u^{-1}(t)B^T(t) - \gamma^{-2}F_w(t)R_w^{-1}(t)F_w^T(t)\right)Q(t), \\ Q(T) = G_f, \end{cases} \tag{86}$$

Optimal H^∞-control corresponds to the minimum value $\gamma^2_{\min}$ of the parameter γ^2, under which condition (inequality) is satisfied

$$E - \gamma^{-2}Q(t)P(t) > 0,\ t \in [t_0, T], \tag{87}$$

where symmetric positive definite matrices $P(t)$ and $Q(t)$ satisfy, respectively, systems (85) and (86).

We emphasize that systems (85) and (86) are being resolved independently in the forward and reverse times. However, the parameter γ^2 cannot be selected arbitrarily. It must satisfy the condition $\gamma^2 > \gamma^2_{\min}$. Otherwise, the matrices $P(t)$ and $Q(t)$ become unbounded. Note that the values $\gamma^2_{\min}$ can be found numerically, for example, by the method of half division of a segment.

Note also that this $\gamma^2_{\min}$ is the minimum value of criterion (3) for the most adverse disturbances acting on the system and in the observation channel. In this case, the worst perturbations are determined by the ratios

$$x_0 = \left(E - \gamma^{-2}R_0^{-1}F_0^TQ(t_0)F_0\right)^{-1}\hat{x}_0, \tag{88}$$

$$w(t) = \gamma^{-2}R_w^{-1}(t)F_w^T(t)\left(E - \gamma^{-2}Q(t)P(t)\right)^{-1}Q(t)\hat{x}(t),\ v(t) = 0. \tag{89}$$

Note.

Using coordinate conversion

$$x_c(t) = \left(E - \gamma^{-2}P(t)Q(t)\right)^{-1}\hat{x}(t) \tag{90}$$

optimal H^∞-control can be represented as

$$u(t) = -G_u^{-1}(t)B^T(t)Q(t)x_c(t), \tag{91}$$

where $x_c(t)$ is the output of the compensator

$$\begin{cases} \dfrac{dx_c(t)}{dt} = Ax_c(t) + Bu(t) + \gamma^{-2} F_w R_w^{-1} F_w^T Q x_c(t) + \\ \qquad + \left(E - \gamma^{-2} PQ\right)^{-1} PC^T R\left(y(t) - Cx_c(t)\right), \\ x_c(t_0) = \left(E - \gamma^{-2} F_0 R_0^{-1} F_0^T Q(t_0)\right)^{-1} F_0 \hat{x}_0, \end{cases} \tag{92}$$

or

$$\begin{cases} \dfrac{dx_c(t)}{dt} = A_c(t) x_c(t) + B_c(t) y(t), \\ x_c(t_0) = x_c^0, \end{cases} \tag{93}$$

in which is indicated

$$A_c(t) = A(t) - B(t) G_u^{-1}(t) B^T(t) Q(t) + \gamma^{-2} F_w(t) R_w^{-1}(t) F_w^T(t) Q(t) - \\ - \left(E - \gamma^{-2} P(t) Q(t)\right)^{-1} P(t) C^T(t) R(t) C(t), \tag{94}$$

$$B_c(t) = \left(E - \gamma^{-2} P(t) Q(t)\right)^{-1} P(t) C^T(t) R(t), \tag{95}$$

$$x_c^0 = \left(E - \gamma^{-2} F_0 R_0^{-1} F_0^T Q(t_0)\right)^{-1} F_0 \hat{x}_0. \tag{96}$$

In this case, the worst perturbations acting on the system are determined by the ratio

$$w(t) = \gamma^{-2} R_w^{-1}(t) F_w^T(t) Q(t) x_c(t). \tag{97}$$

Let us now consider the case of solving the problem of a stationary system (1) at an infinite time interval.

H^∞-optimal control of linear stationary systems at infinite adjustment time. Consider a stationary system (1)

$$\begin{cases} \dfrac{dx(t)}{dt} = Ax(t) + Bu(t) + F_w w(t), \qquad t_0 < t < \infty, \\ x(t_0) = F_0 x_0, \end{cases} \tag{98}$$

in the equation of observation

$$y(t) = Cx(t) + F_v v(t), \tag{99}$$

and quality criteria

$$I(u) = \int_{t_0}^{\infty} \left(x^T(t) G_x x(t) + u^T(t) G_u u(t)\right) dt, \tag{100}$$

With respect to the unknown vector of external perturbations $w(t)$, the vector of measurement errors $v(t)$ and the vector of initial conditions x_0, it is assumed that they belong to the following set of permissible perturbations (uncertainties)

$$\Omega_\xi = \begin{Bmatrix} \xi : \ \xi = (w(t), v(t), x_0), \ w(t) \in L_2(t_0, \infty), \\ v(t) \in L_2(t_0, \infty), \ x_0 \in R^{n_0}; \ \|\xi\|^2 \le 1 \end{Bmatrix}, \tag{101}$$

where the norm $\|\xi\|$ of a vector-valued function is defined by the following expression

$$\| \xi \|^2 = \int_{t_0}^{\infty} \left(w^T(t) R_w w(t) + v^T(t) R_v v(t) \right) dt + \left(x_0 - \hat{x}_0 \right)^T R_0 \left(x_0 - \hat{x}_0 \right). \quad (102)$$

Then, the H^∞-optimal solution to the problem of minimax control

$$\inf_u \sup_{\xi \in \Omega_\xi} I(u) = \gamma^2_{\min}, \quad (103)$$

presentable in the form

$$u(t) = -G_u^{-1} B^T Q x_c(t), \quad (104)$$

where $x_c(t)$ is the output of the compensator

$$\begin{cases} \dfrac{dx_c(t)}{dt} = A_c x_c(t) + B_c y(t), \\ x_c(t_0) = x_c^0, \end{cases} \quad (105)$$

in which is indicated

$$A_c = A - BG_u^{-1} B^T Q + \gamma^{-2} F_w R_w^{-1} F_w^T Q - \left(E - \gamma^{-2} PQ \right)^{-1} PC^T RC, \quad (106)$$

$$B_c = \left(E - \gamma^{-2} PQ \right)^{-1} PC^T R, \quad (107)$$

$$x_c^0 = \left(E - \gamma^{-2} F_0 R_0^{-1} F_0^T Q(t_0) \right)^{-1} F_0 \hat{x}_0, \; R = (F_v^{-1})^T R_v F_v^{-1}. \quad (108)$$

Matrices $P = P^T > 0$ and $Q = Q^T > 0$ are the solutions of the following matrix algebraic Rikatti equations

$$AP + PA^T - P\left(C^T RC - \gamma^{-2} G_x \right) P + F_w R_w^{-1} F_w^T = 0, \quad (109)$$

$$-A^T Q - QA + Q\left(BG_u^{-1} B^T - \gamma^{-2} F_w R_w^{-1} F_w^T \right) Q - G_x = 0, \quad (110)$$

in which the parameter γ^2 must satisfy the condition

$$E - \gamma^{-2} QP > 0. \quad (111)$$

The minimum value $\gamma^2_{\min}$ of the parameter γ^2 under which condition (111) is satisfied corresponds to optimal control.

The worst (most unfavorable) perturbations are thus given by formulas

$$w(t) = \gamma^{-2} R_w^{-1} F_w^T Q x_c(t), \; v(t) = 0, \; x_0 = \left(E - \gamma^{-2} R_0^{-1} F_0^T Q F_0 \right)^{-1} \hat{x}_0. \quad (112)$$

The state vector estimate $\hat{x}(t)$ can be found by the formula

$$\hat{x}(t) = \left(E - \gamma^{-2} PQ \right) x_c(t). \quad (113)$$

Conclusions

Automatic control theory is moving towards complicating the phenomena under study, processes and reducing information about the control system, the object, its features, properties, characteristics, conditions of operation, uncertainties and external influences. Considering all of the above, the chosen area of research is promising and has a high level of relevance.

Thus, the purpose of the article, declared at the beginning of the work, is achieved, the proposed solution of the problem of finding the optimal control as an output feedback, which minimizes the integral-quadratic criterion of operation in the conditions of uncertainty in the most unfavorable perturbations. The results of the studies are presented in the form of practical formulas, according to which the corresponding calculations are acceptable when modeling the control processes in the considered linear dynamic non-stationary object with uncertainties.

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