SOLUTION OF HEAT EQUATION STATIONARY PROBLEM

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Key words:	ABSTRACT
MathCAD **Partial derivatives** **Heat-transfer equation** **Built-in functions**	This paper aims to propose the use of information technology in solving problems of mathematical physics. The paper describes the software implementation of the solution for heat equation with the use of MathCAD mathematical package. Benefits of using information technology in solving different problems of mathematical physics were analyzed in comparison with traditional methods during university course. The specificity of combination of mathematical physics problems and built-in functions of MathCAD environment makes it possible to operate with no cumbersome expressions. Promising area of future research is methodical, mathematical and algorithmic aspects of efficient software implementations of numerical methods within MathCAD environmemt.
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This paper aims to propose the use of information technology in solving problems of mathematical physics. The paper describes the software implementation of the solution for heat equation with the use of MathCAD mathematical package. Benefits of using information technology in solving different problems of mathematical physics were analyzed in comparison with traditional methods during university course. The specificity of combination of mathematical physics problems and built-in functions of MathCAD environment makes it possible to operate with no cumbersome expressions. Promising area of future research is methodical, mathematical and algorithmic aspects of efficient software implementations of numerical methods within MathCAD environmemt.

РОЗВ'ЯЗОК СТАЦІОНАРНОЇ ЗАДАЧІ РІВНЯННЯ ТЕПЛОПРОВІДНОСТІ

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У статті розглянуто програмні реалізації розв'язку рівняння теплопровідності за допомогою математичного пакета MathCAD. Проаналізовано переваги використання інформаційних технологій у процесі розв'язку задач математичної фізики порівняно з традиційними методами під час навчання студентів у ВНЗ. Специфіка поєднання у процесі розв'язання задач математичної фізики і вбудованих функцій середовища MathCAD надає можливість уникнути громіздких виразів. Перспективним напрямом подальших наукових досліджень є методичні, математичні й алгоритмічні аспекти створення ефективних програмних реалізацій чисельних методів у середовищі MathCAD.

Ключові слова: MathCAD, частинні похідні, рівняння теплопровідності, вбудовані функції.

Introduction. Mathematical package MathCAD is popular among mathematicians-non-programmers first of all because of its WYSIWYG — interface (e.i. "What You See Is What You Get") that best approximates the look of traditional Mathcad instruments to calculations "on paper".

MathCAD package is one of the widespread packages containing "the math of an engineering university" and provides the user with a great set of tools for

implementation of many methods of solving mathematical tasks with the help of the computer.

As a rule, the process of solving partial derivatives equations is a tedious scientific work that requires extensive training of the researcher. Working on solving equations often starts with examination of greatly simplified cases in order to develop the algorithm for solving the problem and figure out its features in terms of computational experiments. Here you may fully experience the benefits of MathCAD package — its simplicity, intuitive WYSIWYG-interface and using by nonprofessionals. Solving the heattransfer problem using MathCAD package [2] is illustrated in this article on a simple example.

The purpose of this paper is to prove expediency of using built-in functions of MathCAD package in solving the tasks of mathematical physics.

Differential equations in partial derivatives represent one of the most challenging and at the same time interesting tasks of computational mathematics. These equations are characterized by the fact that there is no single universal algorithm of solving them.

One can successfully simulate the most complex phenomena and processes by equations in partial derivatives (diffusion, hydrodynamics, quantum mechanics, ecology, etc.).

Different equations in partial derivatives require finding not one function, as for tolerance range, but several variables, for example, $f(x,y)$ or $f(x,t)$. Formulation of such problems include the equation (or system of equations) containing derivatives of the unknown function on different variables (partial derivatives), as well as a number of boundary conditions on the boundaries of the computational domain [1].

The Research Methods. A heat equation is considered as an example. A special case of the heat equation determines the stationary one that is time-dependent problem. A stationary equation describes physical picture of temperature distribution on a plate that does not change over time. This picture can be provided under condition that the stationary heat source operates for a long time, and transients caused by switching it stopped. If the function is independent of time, then the time derivative on the left side of the equation is zero, and the equation can be written as follows:

$$\frac{\partial^2}{\partial^2 x^2} T(x,y) + \frac{\partial^2}{\partial^2 y^2} T(x,y) = -f(x,y). \tag{1}$$

Poisson's equation is most often used to describe some stationary processes, for example, temperature distribution over an area around the source (i.e. stationary process of heat distribution). In MathCAD system solution of Poisson's equation is sought in nods of dimension grid of points $(M+0) \times (M+1)$ (where $M=2n$) at the corners of which the necessary boundary conditions must be given. If the heat flow is completely dissipated by the environment, they should be zero. There is a special function multigrid $(F, ncycle)$ in MathCAD for description of this most common situation.

In the function:

- F — matrix of dimension $(M+0) \times (M+1)$ (*where* $M=2n$), which defines the right side of Poisson's equation. In case of description of stationary process of heat

distribution it is necessary to determine position and strength of the source (or absorber) of heat in the function. It is necessary to specify the initial conditions in the edge elements of matrix F (this way the system is given dimension of grid solution).

- *ncycle* — this parameter determines the number of cycles, that are used by numerical algorithm for each iteration. In most cases it is determined as 2 [4].

An example of solution of stationary equation of heat conductivity is below. Please note that the matrix of right-hand side of Poisson's equation F must be defined with the opposite sign.

Solution of Poisson's equation when there is one source using MathCAD and function multigrid

- set dimension of approximating grid:

$$M := 2^5 ;$$

- determine the boundary conditions:

$$F1_{M,M} := 0 ;$$

- set coordinates power of the source and its power:

$$X := 13, Y := 13, Power := 12300$$

$$F1_{X,Y} := Power .$$

Poisson equation can be solved using the multigrid as follows:

$$T1 := multirig(-F1, 2) .$$

Heat eqution stationary problem solution is presented in Table 1:

Table 1. **Solution of Poisson's equation when there is one source**

		0	1	2	3
	0	0	0	0	0
	1	0	0,021	0,042	0,063
	2	0	0,042	0,084	0,127
	3	0	0,063	0,127	0,19
	4	0	0,084	0,169	0,254
T1 =	5	0	0,105	0,211	0,317
	6	0	0,126	0,252	0.38
	7	0	0,145	0,292	0,441
	8	0	0,164	0,329	0,498
	9	0	0,18	0,363	0,55
	10	0	0,194	0,391	0,594
	11	0	0,205	0,414	0,63
	12	0	0,213	0,43	...

Solution of Poisson's equation from one source is visualized at Figure 1 (a) — surface plot, b) — counter plot).

By means of *multigrid* function one can simulate the situation when several sources and absorber of heat locate on one area. For this it is necessary to determine the needed elements of the matrix F.
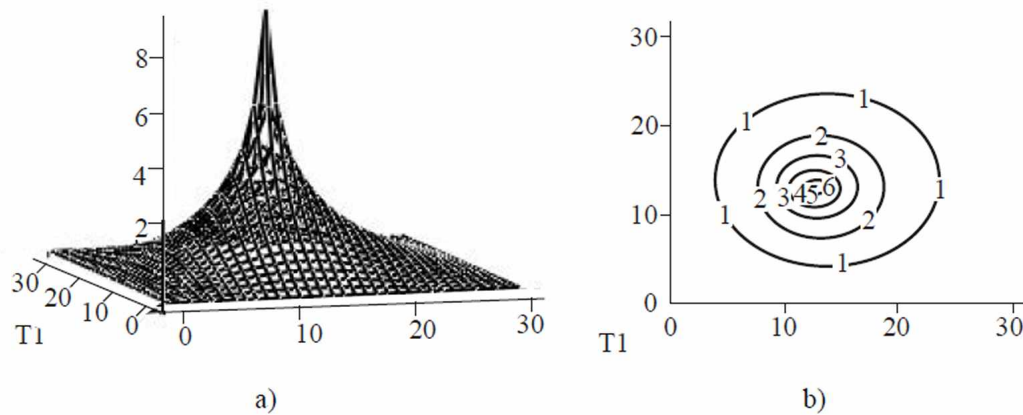
Fig. 1. Poisson's equation solution with one source given

In cases when the Poisson equation has non-zero values of the boundary conditions, its solution can be obtained with the help of special built-in function *relax (a, b, c, d, e, F, v, rjac)*, where

- *a, b, c, d, e* — square matrices of equal dimension containing coefficients of approximating differential equation of five point diversity scheme.

- *F* — square matrix that corresponds to the right side of the Poisson equation;

- *V* — square matrix containing the boundary conditions and initial approximations for solution;

- *rjac* — the spectral radius of the Jacobi iteration. It can take values from 0 to 1 and is needed to characterize the rate of convergence of iterations.

The function *relax* uses relaxation algorithm (refers to methods of unconstrained minimization of functions of several variables) for solving equations of mathematical physics [4].

In fact, this feature can be used to solve elliptic equations of general form:

$$A\frac{\partial^2 u}{\partial^2 x^2} + 2B\frac{\partial^2 u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial^2 y^2} + a\frac{\partial u}{\partial x} + b\frac{\partial u}{\partial y} + cu = F(x,y) \qquad (2)$$

$$D = AC - B^2 > 0,$$

which can be reduced to an equation in finite differences

$$a_i, \mu_{i+1,j} + b_i, \mu_{i-1,j} + c_i, \mu_{i,j+1} + d_i, \mu_{i,j-1} + e_i, \mu_{i,j} = f_{i,j}.$$

In partial case for the Poisson equation coefficients

$$a_{i,j} = b_{i,j} = c_{i,j} = d_{i,j} = 1, e_{i,j} = -1.$$

The idea of relaxation method is as follows. If there are no sources (Laplace equation), then the value function in this node at the current step $k+1$ is defined as the average value of the function at the nearest nodes in the previous step k [3].

Relaxation method converges rather slowly, as in fact it uses the difference scheme with the best possible step for two-dimensional case $\tau = \frac{h^2}{4}$.

Using relaxation method one has to set an initial approximation, i.e. the function value for all nodes of the field and boundary conditions.

Relax function returns a square matrix, wherein: location of an element in the matrix corresponds to its position inside the square matrix; this value approximates the solution at this point.

Relax function is used when value of the unknown function *U(x,y)* are known on all four sides of the square area.

Here is a sample of solution of Poisson's equation using the scheme of approximation of the "cross" type.

The solution of the Poisson equation using functions *multigrid* and *relax* should be considered. For example, there are two sources of energy with a capacity of 150 units and 300 units. Coordinates of the energy sources: the first source is in the middle, the second is twice shifted from the first and works in negative coordinate systems. Results should be presented in a surface plot.

Solution of Poisson's equation having several heat source and absorbers using MathCAD and relax function

The approximating matrix should be set:

$$M := 2^5$$

$$i := 0..M \quad j := 0..M$$

$$a_{i,j} := 1$$

$$b := a, \ e1 := -4 \cdot a, \ c1 := a, \ d := a$$

$$F1_{M,M} := 0$$

$$F1_{23,12} := 23 \quad F1_{13,13} := -29$$

$$v_{i,j} := 0$$

$$Z := relax(a,b,c1,d,e1,-F1,v,0,5) \ .$$

Graphic solution of Poissons's equation with several heat sources and absorbents given with the use of relax function (a) — surface plot, b) — counter plot) is presented at Figure 2.
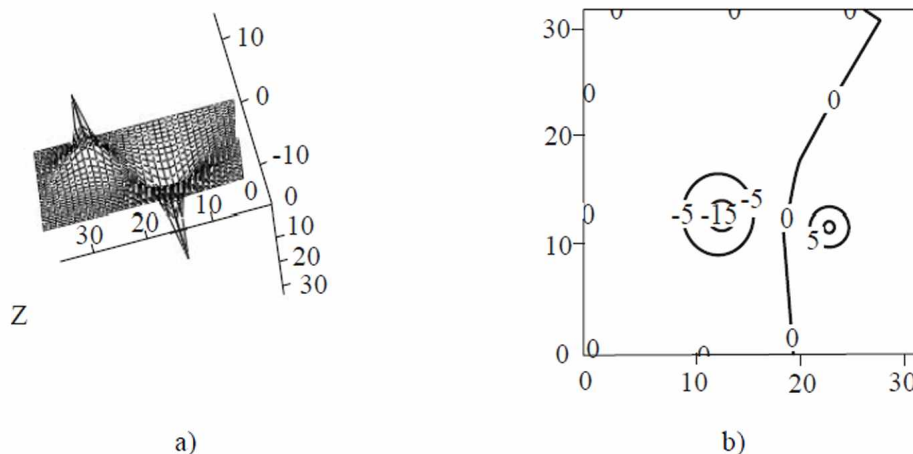

a)   b)

Fig. 2. Graphic solution of Poisson's equation with the use of relax function

Solution of Poisson's equation having several heat sources and absorbers using MathCAD and multigrid function.

$$R1 := 2^5 \quad I_{R1,R1} := 0$$

$$I_{\frac{R1}{2},\frac{R1}{2}} := 100 \quad I_{R1-2,\frac{R1}{2}} := -165$$

$$I_{R1 \cdot \frac{3}{4}, R1 \cdot \frac{3}{4}} := 150 \quad I_{R1 \cdot \frac{3}{4}, R1 \cdot \frac{2}{4}} := 50$$

$$T2 := multirid(-1, 2)_.$$

Graphic solution of Poissons's equation with several heat sources and absorbents given with the use of MathCad multigrid function (a) — surface plot, b) — counter plot) is presented at Figure 3.
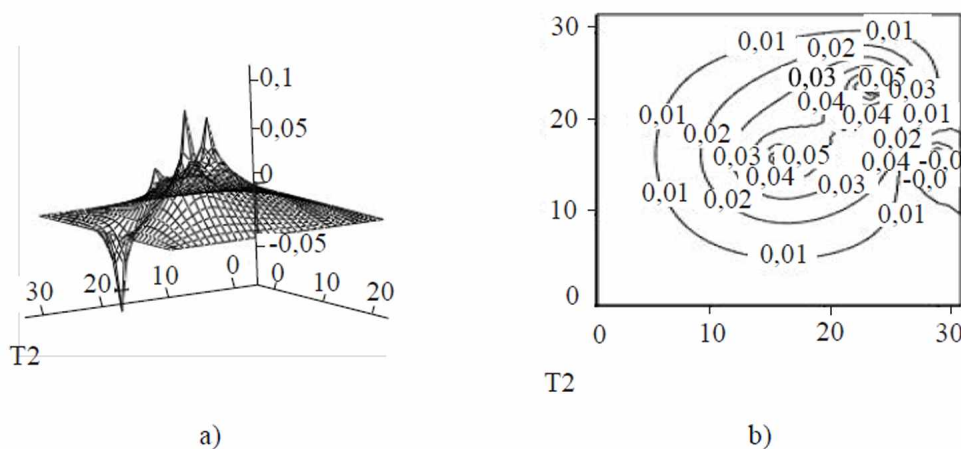

a)   b)

Fig. 3. Graphic solution of Poissons's equation with the use of MathCad multigrid function

This way the built-in MathCAD functions allows to solve mathematical physics tasks.

Conclusions

This paper gives an example of using the differential equations for modeling such real process as thermal conductivity. Numerical and empirical methods are often used for solving direct and inverse tasks in many fields of physics, mathematics and other natural sciences. The role of differential equations in solving of such tasks should be noted, as it is not always possible to determine the functional relationship between searched and given variables, but one can establish the differential equation that allows to predict precisely course of a certain process under certain conditions.

When using MathCAD, the numerical solution of differential equations in partial derivatives can be easily found.

Solution of heat conductivity equation was found by means of built-in functions of MathCAD package, what allowed to avoid operation with cumbersome expressions.

Solving task using MathCAD gives the user an opportunity to enhance the intellectual activity, ability to automate calculations, and to visualize the results of the calculations due to powerful graphic.

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РЕШЕНИЕ СТАЦИОНАРНОЙ ЗАДАЧИ УРАВНЕНИЯ ТЕПЛОПРОВОДНОСТИ

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В статье рассмотрены программные реализации решения уравнения теплопроводности с помощью математического пакета MathCAD. Проанализированы преимущества использования информационных технологий в процессе решения задач математической физики в сравнении с традиционными методами во время обучения студентов в вузе. Специфика сочетания в процессе решения задач математической физики и встроенных функций среды MathCAD дает возможность избежать громоздких выражений. Перспективным направлением дальнейших научных исследований являются методические, математические и алгоритмические аспекты создания эффективных программных реализаций численных методов в среде MathCAD.

Ключевые слова: MathCAD, частные производные, уравнения теплопроводности, встроенные функции.