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ON BELONGING OF NAFTALEVICH-TSUJI PRODUCTS TO
A GENERALIZED CONVERGENCE CLASS

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In terms of the zeros distribution, a class of Naftalevich-Tsuji products π defined by the convergence of the integral $\int_0^1 \frac{\alpha(\ln^+ M_\pi(r))}{\beta(1/(1-r))} dr$ is described, where α and β are positive continuous increasing to $+\infty$ functions.

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В терминах распределения нулей описан класс произведений Нафтаlevича-Цудзи π , определенный сходимостью интеграла $\int_0^1 \frac{\alpha(\ln^+ M_\pi(r))}{\beta(1/(1-r))} dr$, где α и β — положительные непрерывные возрастающие к $+\infty$ функции.

1. Introduction. For an analytic in $\mathbb{D}_R = \{z : |z| < R\}$, $0 < R \leq +\infty$, function f let $M_f(r) = \max\{|f(z)| : |z| = r < R\}$, and let z_k be the zeros of f lying in $\mathbb{D}_R$. G. Valiron [1] defined for entire functions ($R = +\infty$) of order $\rho \in (0, +\infty)$ the convergence class by the condition $\int_1^\infty \ln M_f(r) r^{-\rho-1} dr < +\infty$ and proved that if f belongs to this convergence class then $\sum_{k=1}^\infty |z_k|^{-\rho} < +\infty$. If ρ is a noninteger number then the last condition is also sufficient in order that f belongs to the convergence class.

Let L be a class of positive continuous increasing to $+\infty$ functions on $[x_0, +\infty)$. For canonical products of infinite order in [2] the following theorem is proved.

Theorem A. Let a continuously differentiable function $\alpha \in L$ be such that $\alpha'(x)x \ln x \ln \ln x = O(1)$ as $x \rightarrow +\infty$, $\alpha'(x)e^{\alpha(x)}$ is a nonincreasing function on $[x_0, +\infty)$ and $\alpha'(x-1)e^{\alpha(x-1)} = O(\alpha'(x)e^{\alpha(x)})$ as $x \rightarrow +\infty$.

In order that a canonical product $\pi(z) = \prod_{n=1}^\infty E(z/z_n, [\ln n])$ belongs to the convergence α -class (i.e. $\int_{r_0}^\infty \exp\{\alpha(\ln M_f(r))\} r^{-\rho-1} dr < +\infty$) it is necessary and sufficient that

$$\sum_{k=k_0}^\infty \frac{\alpha'(k) \exp\{\alpha(k)\}}{|z_k|^2} < +\infty.$$

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We remark that in the proof of Theorem A in [2] the condition $\lim_{r \rightarrow +\infty} \ln n(r)/\ln r > 0$ is used, where $n(r) = \sum_{|z_k| \leq r} 1$ is the counting function of the sequence $\{z_k\}$, but this condition is unnecessary.

The following theorem is proved in [3].

Theorem B. Let $\alpha \in L$, $\beta \in L$, $\omega \in L$, $\alpha(x^2) = O(\alpha(x))$ and $\beta(2x) = O(\beta(x))$ as $x \rightarrow +\infty$, $\int_{x_0}^{\infty} \frac{dx}{\beta(x)} < +\infty$ and $\ln n(r) \sim \omega(r) \ln r$ as $r \rightarrow +\infty$.

In order that a canonical product $\pi(z) = \prod_{n=1}^{\infty} E(z/z_n, [2\omega(|z_n|)])$ belongs to the convergence $\alpha\beta$ -class, i.e.

$$\int_{r_0}^{\infty} \frac{\alpha(\ln M_f(r))}{r\beta(\ln r)} dr < +\infty,$$

it is necessary and sufficient that

$$\sum_{k=k_0}^{\infty} (\alpha(k) - \alpha(k-1)) B(\ln |z_k|) < +\infty,$$

where $B(x) = \int_x^{\infty} \frac{dt}{\beta(t)}$.

Here we obtain analogues of Theorem A and B for Naftalevich-Tsuji canonical products.

So, let (a_k) be a sequence of numbers from $\mathbb{D} = \mathbb{D}_1$ such that $|a_k| \leq |a_{k+1}|$, $\sum_{k=1}^{\infty} (1 - |a_k|)^p = +\infty$ and $\sum_{k=1}^{\infty} (1 - |a_k|)^{p+1} < \infty$. Studying a factorization of meromorphic functions in $\mathbb{D}$, A. G. Naftalevich [4-5] and M. Tsuji [6] constructed and used the following canonical product

$$\pi(z) = \prod_{k=1}^{\infty} E\left(\frac{1 - |a_k|^2}{1 - \bar{a}_k z}, p\right), \quad (1)$$

where

$$E(u, p) = (1 - u) \exp \left\{ u + \frac{u^2}{2} + \dots + \frac{u^p}{p} \right\}$$

is a primary Weierstrass factor. Product (1) is absolutely and uniformly convergent on each compact set from $\mathbb{D}$ and, thus, π is an analytic function in $\mathbb{D}$. A. G. Naftalevich [5] proved that if p is a noninteger number then product (1) belongs to a convergence class (i.e. $\int_0^1 (1-r)^{p-1} T(r, \pi) dr < +\infty$, where $T(r, \pi)$ is the Nevanlinna characteristic) if and only if $\sum_{k=1}^{\infty} (1 - |a_k|)^p < +\infty$.

We construct Naftalevich-Tsuji products which belong to a generalized convergence class in the case when $\sum_{k=1}^{\infty} (1 - |a_k|)^p = +\infty$ for each $p \in \mathbb{N}$.

2. Auxiliary results. We use the Dirichlet series theory. Let $0 < \lambda_n \nearrow \infty$ ($n \rightarrow \infty$) and a Dirichlet series

$$F(\sigma) = \sum_{n=1}^{\infty} a_n \exp\{\sigma \lambda_n\} \quad (a_n > 0) \quad (2)$$

be convergent for all $\sigma \in \mathbb{R}$. Let $\mu(\sigma) = \max\{a_n \exp\{\sigma \lambda_n\} : n \geq 1\}$ be the maximal term and $\nu(\sigma) = \max\{n : a_n \exp\{\sigma \lambda_n\} = \mu(\sigma)\}$ be the central index of series (2). The following lemma is well known [7; 8, p. 23] in the case when $\lambda_n \uparrow \infty (n \rightarrow \infty)$. In virtue of simplicity we give its proof.

Lemma 1. *If $\overline{\lim}_{n \rightarrow \infty} \frac{\ln n}{\ln(1/a_n)} = h < 1$, then $F(\sigma) \leq K(\varepsilon) \mu\left(\frac{\sigma}{1-h-\varepsilon}\right)$ for every $\varepsilon \in (0, 1-h)$ and all $\sigma \geq \sigma_0(\varepsilon)$, where $K(\varepsilon) = \text{const} > 0$.*

Proof. Since $\sum_{n=1}^{\infty} a_n \exp\{\sigma \lambda_n\}$ is convergent for all $\sigma \in \mathbb{R}$, we have $\frac{1}{\lambda_n} \ln \frac{1}{a_n} \rightarrow +\infty (n \rightarrow \infty)$. We put

$$N_1 = \left\{ n : \frac{1}{\lambda_n} \ln \frac{1}{a_n} \leq \frac{\sigma}{1-h-\varepsilon} \right\} \text{ and}$$

$$N_2 = \left\{ n : \frac{1}{\lambda_n} \ln \frac{1}{a_n} > \frac{\sigma}{1-h-\varepsilon} \right\}.$$

Then

$$\begin{aligned} F(\sigma) &= \sum_{n \in N_1} a_n \exp\left\{ \frac{\sigma \lambda_n}{1-h-\varepsilon} \right\} \exp\left\{ -\frac{(h+\varepsilon)\sigma \lambda_n}{1-h-\varepsilon} \right\} + \sum_{n \in N_2} a_n \exp\{\sigma \lambda_n\} \leq \\ &\leq \mu\left(\frac{\sigma}{1-h-\varepsilon}\right) \sum_{n \in N_1} \exp\left\{ -\frac{(h+\varepsilon)\lambda_n}{1-h-\varepsilon} \frac{1-h-\varepsilon}{\lambda_n} \ln \frac{1}{a_n} \right\} + \\ &\quad + \sum_{n \in N_2} a_n \exp\left\{ \lambda_n \frac{1-h-\varepsilon}{\lambda_n} \ln \frac{1}{a_n} \right\} \leq \\ &\leq 2\mu\left(\frac{\sigma}{1-h-\varepsilon}\right) \sum_{n=1}^{\infty} \exp\left\{ -(h+\varepsilon) \ln \frac{1}{a_n} \right\}. \end{aligned}$$

Since $\ln n \leq (h+\varepsilon/2) \ln(1/a_n)$, $n \geq n_0(\varepsilon)$, we obtain the required inequality. $\square$

Lemma 2. *For all $\sigma > \sigma_0$*

$$\ln \mu(\sigma) = \ln \mu(\sigma_0) + \int_{\sigma_0}^{\sigma} \lambda_{\nu(t)} dt.$$

In the case when $\lambda_n \uparrow \infty (n \rightarrow \infty)$ Lemma 2 is well known [9, p. 184; 8, p. 17]. For the case when $\lambda_n \nearrow \infty (n \rightarrow \infty)$ its proof is the same as in [8].

We need also the following

Lemma 3. [10, p.18] *The inequality*

$$\ln |E(z, p)| \leq 3e(2 + \ln p) |z|^{p+1} / (1 + |z|)$$

holds for all $z \in \mathbb{C}$.

Now, let $z_k \in \mathbb{D}$, $0 < |z_k| \nearrow 1$ ($k \rightarrow \infty$) and (p_k) be a sequence of positive integers, $1 \leq p_k \nearrow \infty$ ($k \rightarrow \infty$). Consider the canonical product

$$\pi(z) = \prod_{k=1}^{\infty} E\left(\frac{1 - |z_k|^2}{1 - \bar{z}_k z}, p_k\right), \quad |z| = r \in [0, 1]. \quad (3)$$

In view of Lemma 3

$$\begin{aligned} \ln |\pi(z)| &= \sum_{k=1}^{\infty} \ln \left| E\left(\frac{1 - |z_k|^2}{1 - \bar{z}_k z}, p_k\right) \right| \leq \\ &\leq \sum_{k=1}^{\infty} 3e(2 + \ln p_k) \frac{((1 - |z_k|^2)/|1 - \bar{z}_k z|)^{p_k+1}}{1 + (1 - |z_k|^2)/|1 - \bar{z}_k z|} \leq \end{aligned} \quad (4)$$

$$\leq \sum_{k=1}^{\infty} \left(\frac{K_1(1 - |z_k|)}{|1 - \bar{z}_k z|} \right)^{p_k+1} \leq \sum_{k=1}^{\infty} \left(\frac{K_1(1 - |z_k|)}{1 - r} \right)^{p_k+1}, \quad K_1 > 1. \quad (5)$$

Here and further by K_j we denote positive constants.

It is easy to see that if

$$\lim_{n \rightarrow \infty} \frac{p_k}{\ln k} \ln \frac{1}{1 - |z_k|} > 1 \quad (6)$$

then the last series in (4) is convergent and, thus, π is an analytic function in $\mathbb{D}$. Condition (5) holds if, for example, $p_k = [\ln k]$.

For $\alpha \in L$ and $\beta \in L$ we say that π belongs to the convergence $\alpha\beta$ -class if

$$\int_{r_0}^1 \frac{\alpha(\ln M_{\pi}(r))}{(1-r)\beta(1/(1-r))} dr < +\infty. \quad (7)$$

Proposition 1. Let $\alpha \in L$, $\beta \in L$, $\alpha(x^2) = O(\alpha(x))$ and $\beta(2x) = O(\beta(x))$ as $x \rightarrow +\infty$ and $\int_{r_0}^1 \frac{\alpha(x)dx}{x\beta(x)} < +\infty$. Then (6) implies

$$\int_{r_0}^1 \frac{\alpha(n(r))}{(1-r)\beta(1/(1-r))} dr < +\infty. \quad (8)$$

Proof. Indeed, by the Jensen inequality

$$\begin{aligned} \ln M_{\pi}\left(\frac{1+r}{2}\right) &\geq \int_0^{(1+r)/2} \frac{n(t)}{t} dt + \ln |\pi(0)| \geq \int_r^{(1+r)/2} \frac{n(t)}{t} dt + \ln |\pi(0)| \geq \\ &\geq \frac{2n(r)}{1+r} \int_r^{(1+r)/2} dt + \ln |\pi(0)| = \frac{1-r}{1+r} n(r) + \ln |\pi(0)|, \end{aligned}$$

that is

$$n(r) \leq \frac{3}{1-r} \ln M_{\pi}\left(\frac{1+r}{2}\right), \quad r \in [r_0, 1).$$

Therefore, in view of the condition $\alpha(x^2) = O(\alpha(x))$ ($x \rightarrow +\infty$) we have

$$\begin{aligned}
 \alpha(n(r)) &= \alpha \left(\exp \left\{ \ln \frac{3}{1-r} + \ln \ln M_\pi \left(\frac{1+r}{2} \right) \right\} \right) \leq \\
 &\leq \alpha \left(\exp \left\{ 2 \max \left\{ \ln \frac{3}{1-r}, \ln \ln M_\pi \left(\frac{1+r}{2} \right) \right\} \right\} \right) \leq \\
 &\leq K_2 \alpha \left(\exp \left\{ \max \left\{ \ln \frac{3}{1-r}, \ln \ln M_\pi \left(\frac{1+r}{2} \right) \right\} \right\} \right) = \\
 &= K_2 \max \left\{ \alpha \left(\frac{3}{1-r} \right), \alpha \left(\ln M_\pi \left(\frac{1+r}{2} \right) \right) \right\} \leq \\
 &\leq K_2 \left(\alpha \left(\frac{3}{1-r} \right) + \alpha \left(\ln M_\pi \left(\frac{1+r}{2} \right) \right) \right). \tag{9}
 \end{aligned}$$

Since $\alpha(3x) = O(\alpha(x))$ ($x \rightarrow +\infty$), the condition $\int_{x_0}^{\infty} \frac{\alpha(x)}{x\beta(x)} dx < +\infty$ implies

$$\int_{r_0}^1 \frac{\alpha(3/(1-r))}{(1-r)\beta(1/(1-r))} dr < +\infty.$$

Clearly,

$$\int_{r_0}^1 \frac{\alpha(\ln M_\pi((1+r)/2))}{(1-r)\beta(1/(1-r))} dr = \int_{(1+r_0)/2}^1 \frac{\alpha(\ln M_\pi(r))}{(1-r)\beta(1/2(1-r))} dr.$$

Therefore, in view of the condition and $\beta(2x) = O(\beta(x))$ ($x \rightarrow +\infty$), (6) and (8) imply (7). $\square$

Proposition 2. Let $\alpha \in L$, $\beta \in L$, $\int_{x_0}^{\infty} \frac{\alpha(x)dx}{x\beta(x)} < +\infty$ and $B(x) = \int_x^{\infty} \frac{dt}{t\beta(t)}$. Then (6) holds if and only if

$$\sum_{n=n_0}^{\infty} (\alpha(n) - \alpha(n-1)) B \left(\frac{1}{1-|z_n|} \right) < +\infty. \tag{10}$$

Proof. Indeed,

$$\begin{aligned}
 &\int_{r_0}^1 \frac{\alpha(n(r))}{(1-r)\beta(1/(1-r))} dr = K_3 + \sum_{n=n_0}^{\infty} \int_{|z_n|}^{|z_{n+1}|} \frac{\alpha(n(r))}{(1-r)\beta(1/(1-r))} dr = \\
 &= K_3 + \sum_{n=n_0}^{\infty} \alpha(n) \int_{|z_n|}^{|z_{n+1}|} \frac{dr}{(1-r)\beta(1/(1-r))} = K_3 + \sum_{n=n_0}^{\infty} \alpha(n) \int_{1/(1-|z_n|)}^{1/(1-|z_{n+1}|)} \frac{dt}{t\beta(t)} =
 \end{aligned}$$

$$\begin{aligned}
&= K_3 + \sum_{n=n_0}^{\infty} \alpha(n) \left(B \left(\frac{1}{1-|z_n|} \right) - B \left(\frac{1}{1-|z_{n+1}|} \right) \right) = \\
&= K_4 + \sum_{n=n_0}^{\infty} (\alpha(n) - \alpha(n-1)) B \left(\frac{1}{1-|z_n|} \right).
\end{aligned}$$

□

Proposition 3. If $p_k = [\ln k]$ then there exist constants $K_2 > 1$ and $K_3 > 1$ such that for canonical product (3)

$$\ln \ln M_{\pi}(r) \leq K_3 \int_0^{1-(1-r)/K_2} \frac{\ln n(t)}{1-t} dt. \quad (11)$$

Proof. We put $\sigma = \ln \frac{1}{1-r}$. Then for $\eta > 0$ we obtain from (4)

$$\begin{aligned}
\ln M_{\pi}(r) &\leq \sum_{n=1}^{\infty} (K_1(1-|z_n|)e^{\sigma})^{[\ln n]+1} = \\
&= \sum_{n=1}^{\infty} (K_1(1-|z_n|)e^{\sigma+1+\eta})^{[\ln n]+1} e^{-(1+\eta)([\ln n]+1)} \leq \\
&\leq \mu(\sigma+1+\eta) \sum_{n=1}^{\infty} e^{-(1+\eta) \ln n} \leq \frac{K_4}{\eta} \mu(\sigma+1+\eta),
\end{aligned} \quad (12)$$

where $\mu(\sigma)$ is the maximal term of Dirichlet series (2) with $a_n = (K_1(1-|z_n|))^{[\ln n]+1}$. Since $\mu(\sigma) \rightarrow +\infty$, $\sigma \rightarrow +\infty$, for central index of this series we have $K_1(1-|z_{\nu(\sigma)}|)e^{\sigma} > 1$ that is $|z_{\nu(\sigma)}| \leq 1 - 1/(K_1 e^{\sigma})$, whence $\nu(\sigma) \leq n(1 - 1/(K_1 e^{\sigma}))$. Therefore, by Lemma 2

$$\begin{aligned}
\ln \mu(\sigma) &\leq \ln \mu(0) + \int_0^{\sigma} (\ln \nu(t) + 1) dt \leq \ln \mu(0) + \int_0^{\sigma} (\ln n(1 - 1/(K_1 e^t)) + 1) dt \leq \\
&\leq \ln \mu(0) + \int_0^{1-1/(K_1 e^{\sigma})} \frac{\ln n(t) + 1}{1-t} dt.
\end{aligned} \quad (13)$$

From (11) and (12) we obtain

$$\begin{aligned}
\ln \ln M_{\pi}(r) &\leq \ln \mu \left(\ln \frac{1}{1-r} + 1 + \eta \right) + \ln \frac{K_4}{\eta} \leq \\
&\leq \ln \frac{K_4}{\eta} + K_5 + \int_0^{1-(1-r)/(K_1 e^{1+\eta})} \frac{\ln n(t) + 1}{1-t} dt,
\end{aligned}$$

whence (10) follows. □

Proposition 4. Let $\ln n(r) \sim \omega(r) \ln \frac{1}{1-r}$, $r \uparrow 1$, where ω is a positive continuous decreasing function on $[0, 1)$. If $p_k = 2[\omega(\{z_k\})]$ then there exist constants $K_2 > 1$ and $K_3 > 1$ such that for canonical product (3)

$$\ln \ln M_\pi(r) \leq K_3 \ln n(1 - (1-r)^3/K_2). \quad (14)$$

Proof. Clearly, condition (5) holds and for Dirichlet series (2) with $a_n = (K_1(1-|z_n|))^{p_n+1}$ we have $\lim_{n \rightarrow \infty} \frac{\ln a_n}{\ln(1/a_n)} = \frac{1}{2}$. Therefore, by Lemma 1 from (4) we obtain $\ln M_\pi(r) \leq K_6 \mu(3\sigma)$, $\sigma = \ln \frac{1}{1-r}$. By Lemma 2, $\ln \mu(\sigma) \leq \sigma \lambda_\nu(\sigma)$ for greater σ . As in the proof of Proposition 3, $|z_{\nu(\sigma)}| \leq 1 - 1/(K_1 e^\sigma)$ and, thus,

$$\lambda_\nu(\sigma) \leq p_\nu(\sigma) + 1 \leq 2\omega(|z_{\nu(\sigma)}|) + 1 \leq 2\omega(1 - 1/(K_1 e^\sigma)) + 1 \leq \frac{3}{\sigma + \ln K_1} \ln n(1 - 1/(K_1 e^\sigma)).$$

Therefore,

$$\begin{aligned} \ln \ln M_\pi(r) &\leq \ln K_6 + \ln \mu \left(3 \ln \frac{1}{1-r} \right) \leq \\ &\leq \ln K_6 + 3 \ln \frac{1}{1-r} \frac{\ln n(1 - 1/(K_1 \exp\{3 \ln(1/(1-r))\}))}{\ln(1/(1-r)) + \ln K_1} \leq \\ &\leq \ln K_6 + 3 \ln n(1 - (1-r)^3/K_1), \end{aligned}$$

whence (13) follows. $\square$

3. Main results. First, we consider the case when $p_k = [\ln k]$.

Theorem 1. Let $\alpha \in L$ and $\beta \in L$ be continuously differentiable functions such that

$$\frac{\alpha'(x)}{\alpha(x)} x \ln x \ln \ln x = O(1), \quad \frac{\beta'(x)}{\beta(x)} x = O(1), \quad x \rightarrow +\infty \text{ and } \int_{x_0}^{\infty} \frac{\alpha(x) dx}{x \beta(x)} < +\infty.$$

In order that product (3) with $p_k = [\ln k]$ belongs to the convergence $\alpha\beta$ -class it is necessary and sufficient that its zeros satisfy condition (9).

Proof. From the conditions on α and β it follows that α and β satisfy the conditions of Proposition 1. Therefore, in view of Propositions 1 and 2, we need to prove only that (7) implies (6).

From (10) we obtain

$$\ln \ln M_\pi(r) \leq K_3 \ln n(R) \ln \frac{K_2}{1-r},$$

where $R = 1 - (1-r)/K_2$. We put $\exp_2 x = \exp\{e^x\}$. Then in view of the condition

$\frac{\alpha'(x)}{\alpha(x)} x \ln x \ln \ln x = O(1)$, $x \rightarrow +\infty$, we have

$$\begin{aligned} \alpha(\ln M_\pi(r)) &= \alpha(\exp_2\{\ln \ln \ln M_\pi(r)\}) \leq \\ &\leq \alpha\left(\exp_2\left\{\ln K_3 + \ln \ln \frac{K_2}{1-r} + \ln \ln n(R)\right\}\right) \leq \\ &\leq \alpha\left(\exp_2\left\{3 \max\left\{\ln \ln \frac{1}{1-r}, \ln \ln n(R)\right\}\right\}\right) \leq \\ &\leq K_6 \max\left\{\alpha\left(\frac{1}{1-r}\right), \alpha(n(R))\right\} \leq \\ &\leq K_6 \left(\alpha\left(\frac{1}{1-r}\right) + \alpha(n(R))\right). \end{aligned}$$

Therefore, in view of the conditions $\int_{x_0}^{\infty} \frac{\alpha(x) dx}{x \beta(x)} < +\infty$ and $\frac{\beta'(x)}{\beta(x)} x = O(1)$, $x \rightarrow +\infty$, we obtain

$$\begin{aligned} \int_{r_0}^1 \frac{\alpha(\ln M_\pi(r))}{(1-r)\beta(1/(1-r))} dr &\leq K_7 + K_6 \int_{r_0}^1 \frac{\alpha(n(R))}{(1-r)\beta(1/(1-r))} dr = \\ &= K_7 + K_6 \int_{R_0}^1 \frac{\alpha(n(R)) dR}{K_2(1-R)\beta(1/(K_2(1-R)))} \leq \\ &\leq K_7 + K_8 \int_{R_0}^1 \frac{\alpha(n(R)) dR}{(1-R)\beta(1/(1-R))}, \end{aligned}$$

whence it follows that (7) implies (6). Theorem 1 is proved. $\square$

If we choose $\beta(x) = x^\varrho$, $\varrho > 0$, then $B(x) = 1/(\varrho x^\varrho)$ and from Theorem 2 we obtain

Corollary 1. Let $\alpha \in L$ be a continuously differentiable function, $\frac{\alpha'(x)}{\alpha(x)} x \ln x \ln \ln x = O(1)$ ($x \rightarrow +\infty$). Then for product (3) with $p_k = [\ln k]$

$$\left(\int_{r_0}^1 (1-r)^{\varrho-1} \alpha(\ln M_\pi(r)) dr < +\infty \right) \iff \left(\sum_{n=n_0}^{\infty} (\alpha(n) - \alpha(n-1))(1-|z_n|)^\varrho < +\infty \right).$$

The function $\alpha(x) = \ln x$ does not satisfy the condition of Corollary 1. However, the following result is true.

Theorem 2. For product (3) with $p_k = [\ln k]$

$$\left(\int_{r_0}^1 (1-r)^{\varrho-1} \ln \ln M_\pi(r) dr < +\infty \right) \iff \left(\sum_{n=n_0}^{\infty} \frac{(1-|z_n|)^\varrho}{n} < +\infty \right).$$

Proof. From (10) we obtain

$$\begin{aligned} \int_{r_0}^1 (1-r)^{e-1} \ln \ln M_\pi(r) dr &\leq \frac{K_3}{\varrho} \int_{r_0}^1 \int_0^{1-(1-r)/K_2} \frac{\ln n(t)}{1-t} dt d(-(1-r)^e) \leq \\ &\leq K_4 + K_5 \int_{r_0}^1 (1-r)^{e-1} \ln n(1-(1-r)/K_2) dr = \\ &= K_6 + K_7 \int_{r_0}^1 (1-r)^{e-1} \ln n(r) dr \end{aligned}$$

and, thus, (7) implies (6). In view of Propositions 1 and 2, Theorem 2 is proved. $\square$

Now, we examine the case when $\ln n(r) \sim \omega(r) \ln \frac{1}{1-r}$, $r \uparrow 1$. The following theorem is true.

Theorem 3. Let $\ln n(r) \sim \omega(r) \ln \frac{1}{1-r}$, $r \uparrow 1$, where ω is a positive continuous decreasing function on $[0, 1)$. Let $\alpha \in L$ and $\beta \in L$ be continuously differentiable functions such that $\frac{\alpha'(x)}{\alpha(x)} x \ln x = O(1)$, $\frac{\beta'(x)}{\beta(x)} x \ln x = O(1)$ as $x \rightarrow +\infty$ and $\int_{x_0}^{\infty} \frac{\alpha(x) dx}{x\beta(x)} < +\infty$.

In order that product (3) with $p_k = 2[\omega(|z_k|)]$ belongs to the convergence $\alpha\beta$ -class it is necessary and sufficient that its zeros satisfy condition (9).

Proof. In view of the condition $\frac{\alpha'(x)}{\alpha(x)} x \ln x = O(1)$, $x \rightarrow +\infty$, from (13) we obtain

$$\alpha(\ln M_\pi(r)) \leq K_4 \alpha(n(1-(1-r)^3/K_2))$$

and in view of the condition $\frac{\beta'(x)}{\beta(x)} x \ln x = O(1)$ ($x \rightarrow +\infty$), we have

$$\begin{aligned} \int_{r_0}^1 \frac{\alpha(\ln M_\pi(r))}{(1-r)\beta(1/(1-r))} dr &\leq K_5 \int_{R_0}^1 \frac{\alpha(n(R))}{(1-R)\beta(1/(K_2(1-R)^{1/3}))} dr \leq \\ &\leq K_6 \int_{R_0}^1 \frac{\alpha(n(R))}{(1-R)\beta(1/(1-R))} dr, \end{aligned}$$

that is (7) implies (6). In view of Propositions 1 and 2, Theorem 3 is proved. $\square$

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