Proceedings of International Conference. Mechanika. 2008.

The Torsion of Elastic Medium with Internal Cylindrical Crack

M. Martynenko *, I. Lebedyeva **

**National University of Food Technology, Volodymyrs'ka 68, 01033, Kyiv Ukraine*
** *Kyiv Taras Shevchenko National University, Volodymyrs'ka 66, 01033, Kyiv Ukraine E-mail: Lebedevai@ukr.net*

Abstract

A problem on torsion of an elastic medium weakened by mathematical cut on a part of the cylindrical surface is solved by exact methods of the linear theory of elasticity. The problem is reduced to a system of dual integral equations with respect to the trigonometric functions with one unknown density. Then the Fredholm integral-differential equation is examined. The analytical expressions for the stress intensity factor, the stress components on the cylinder surface outside the cut and the difference between the displacements of the cut surfaces are obtained. Data obtained are applicable to the study of material damage.
KEY WORDS: *internal nonflat crack, mathematical cut, eigenfunction expansions, cylindrical surface, stress-intensity factor.*

1. Introduction

The cracks in materials are the first steps leading to their fracture. Experiment shows that real cracks are nonflat and have curved surfaces [1, 2]. Such cracks could be modeled by cuts on a part of some surface of revolution with its nonzero curvature. The general approach to solving of mixed boundary problems on equilibrium of three-dimensional bodies weakened by mathematical cuts on parts of the second-degree surfaces was already done by us in [3, 4]. In this paper we study the torsion of elastic spatial body with a crack on a part of cylindrical surface. It is an approximate mathematical model to the problem on stressed state of a matrix reinforced by cylinders of strength under partial separation of the matrix from the reinforcement.

2. Statement of the problem

Let us consider an elastic space with a cut on a part of cylindrical surface S ($\rho = \rho_0, |z| \le z_0, 0 \le \varphi \le 2\pi$) under the torsion symmetric about the axis Oz and the plane $z = 0$. Figure1 shows a meridian section of the crack. We divide the space by the surface S into two domains: the inner domain V_1 and external one V_2. Conditions of stresses and displacements fields continuity have to take place on the surface S out of the cut. For each domain the Lame vector equation of equilibrium

$$2\frac{m-1}{m-2}\operatorname{graddiv}\vec{u} - \operatorname{rotrot}\vec{u} = 0 \qquad 1)$$

must be solved. Here $\vec{u}$ is the displacement vector, m is Poisson's number ($m = 1/\nu$, ν is Poisson's ratio). The unknown coefficients are found from the boundary conditions

$$u_1(\rho, z) = u_2(\rho, z)\,, \sigma^{(1)}_{\rho\varphi} = \sigma^{(2)}_{\rho\varphi}\,,$$

$$(\rho = \rho_0\,, |z| > z_0), \sigma^{(1)}_{\rho\varphi} = \sigma^{(2)}_{\rho\varphi} = f(z)\,, (\rho = \rho_0\,, |z| < z_0) \qquad (2)$$

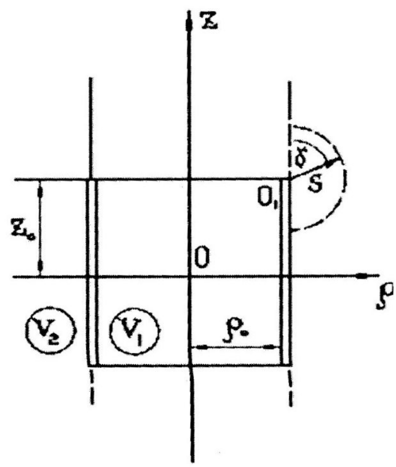

Fig.1 The cylindrical crack in an elastic space

where function $f(z)$ coresponds to the loading transferred on the cut surface according to the superposition principle [3, 4].

3. Method of solution

The study of the problem is reduced to solving of system of dual integral equations with respect to the trigonometric functions with one unknown density. Let us introduce dimensionless variables $\rho / z_0 = s$, $z / z_0 = \xi$, $\lambda\tau^{-1} = z^{-1}$. The components u and $\sigma_{\rho\varphi}$ are the odd functions of variable z, so we represent the stresses and displacements fields by the follows real Fourier integrals

$$Gu_1(s,\xi) = z_0 \int_0^\infty a(\tau) I_1(\tau s) \sin \tau\xi \, d\tau \,, \quad Gu_2(s,\xi) = z_0 \int_0^\infty b(\tau) K_1(\tau s) \sin \tau\xi \, d\tau \,, \tag{3}$$

$$\sigma_{\rho\varphi}^{(1)} = \int_0^\infty \tau a(\tau) I_2(\tau s) \sin \tau\xi \, d\tau \,, \quad \sigma_{\rho\varphi}^{(2)} = -\int_0^\infty \tau b(\tau) K_2(\tau s) \sin \tau\xi \, d\tau \,, \tag{4}$$

$$z_0^2 b(\tau) = b(\tau / z_0) \,, \quad z_0^2 a(\tau) = a(\tau / z_0) \,.$$

The boundary conditions (2) in new variables take the form

$$u_1(s,\xi) = u_2(s,\xi) \,, \ \sigma_{\rho\varphi}^{(1)} = \sigma_{\rho\varphi}^{(2)} \,, \ (s = s_0 = \rho_0 / z_0 \,, \xi > 1), \sigma_{\rho\varphi}^{(1)} = \sigma_{\rho\varphi}^{(2)} = f(\xi, z_0) = g(\xi) \,, \ (s = s_0 \,, \xi < 1) \tag{5}$$

Satisfying the boundary conditions (5), we get a coupled system of dual integral equations

$$z_0 \int_0^\infty \frac{b(\tau)}{\tau s_0 I_2(\tau s_0)} \sin \tau\xi d\tau = 0 \,, (\ \xi > 1) - \int_0^\infty \tau b(\tau) K_2(\tau s_0) \sin \tau\xi_0 d\tau = g(\xi) \,, (\xi < 1) \tag{6}$$

The solution to the system (6) will be in the form

$$b(\tau) = \tau s_0 I_2(\tau s_0) \int_0^1 \varphi(t) J_1(\tau t) \, dt \tag{7}$$

where $J_1(\tau t)$ is a cylindrical Bessel function of the first kind of the first order, $\varphi(t)$ and its derivative are assumed to be continuous for $0 \le t \le 1$. According to equality

$$\int_0^\infty \sin \tau z J_1(\tau t) d\tau = \frac{z}{t} \frac{H(t - z)}{\sqrt{t^2 - z^2}} \tag{8}$$

integral operator (7) satisfies identically the first equation of the system (6). If we substitute (7) into the second equation of the system (6), change the order of integration and use the integral representation for the function $\sin \lambda\xi$ [5]

$$\sin \tau z = \frac{1}{z} \frac{d}{dz} \int_0^z \frac{x^2 J_1(\tau x) dx}{\sqrt{z^2 - x^2}} = \tau \int_0^z \frac{x J_0(\tau x) dx}{\sqrt{z^2 - x^2}} \tag{9}$$

we obtain the following Abel equation

$$\int_0^\xi \frac{u(x) \, dx}{\sqrt{\xi^2 - x^2}} = g(\xi) \,, (0 < \xi < 1) \tag{10}$$

where

$$u(x) = -\int_0^1 \varphi(t) M(t,x) dt \tag{11}$$

$$M(t,x) = \int_0^\infty \tau^3 s_0 I_2(\tau s_0) K_2(\tau s_0) J_0(\tau x) x d\tau \tag{12}$$

Having solved Abel integral equation (10), we get

$$-\int_0^1 \varphi(t) M(t,x) dt = \frac{2}{\pi} \frac{d}{dx} \int_0^x \frac{\xi g(\xi) d\xi}{\sqrt{x^2 - \xi^2}} = F(x) \tag{13}$$

Let us examine improper integral (12). As follows from the properties of Bessel and McDonald functions [5] the integrand does not have singularities on the finite interval. So, convergence of the integral depends on the integrand behavior at high values of parameter τ. The asymptotic formulas for the Bessel and McDonald functions at high values of an argument [5] yield the follows asymptotic dependence for the product of these functions at $\tau >> 1$

$$\tau^3 s_0 I_2(\tau s_0) K_2(\tau s_0) \approx \frac{1}{2} \tau^2 - \frac{15}{16 s_0^2} = a(s_0, \tau) \tag{14}$$

The asymptotic formula (14) allow us to improve the convergence of the integral (12) and present it in the form

$$M(t,x) = -\frac{1}{2}\delta_t'(t-x) - K(t,x) \tag{15}$$

where

$$K(t,x) = \frac{-15}{16s_0^2}\frac{x}{t}H(t-x) + \int_0^\infty \left[\tau^3 s_0 I_2(\tau s_0)K_2(\tau s_0) - a(\tau, s_0)\right] x J_0(\tau x)J_1(\tau t)d\tau \tag{16}$$

After substitution (15) into (13) we get the Fredholm integral-differential equation with respect to the function $\varphi(x)$:

$$\frac{1}{2}\varphi'(x) + \int_0^1 \varphi(t)K(t,x)dt = -F(x) \tag{17}$$

where the regular kernel $K(t,x)$ is determined by formula (16). The condition $\varphi(0) = 0$ completes the Eq. (17).

4. Results

To obtain the stress-tensor component $\sigma_{\rho\varphi}$ on the cylinder surface out of the crack, let us substitute (7) into the second Eq. (4) and use the integral representation (9). If the order of integration is changed, we have

$$\sigma_{\rho\varphi} = -\int_0^\xi \frac{dx}{\sqrt{\xi^2 - x^2}}\left\{\int_0^1 \varphi(t)\left[\int_0^\infty \tau^3 s_0 I_2(\tau s_0)K_2(\tau s_0)xJ_0(\tau x)J_1(\tau t)d\tau\right]dt\right\},\ (\xi > 1) \tag{18}$$

It is easy to see that $\xi = 1$ is a singular point. So, external integral can be represented by sum of the follows integrals

$$\int_0^\xi = \lim_{\varepsilon \to 0}\left[\int_0^{1-\varepsilon} + \int_{1-\varepsilon}^{1+\varepsilon} + \int_{1+\varepsilon}^{\xi}\right] \tag{19}$$

By manipulations of these integrals, we get

$$\sigma_{\rho\varphi} = \frac{1}{2}\frac{\varphi(1)}{\sqrt{\xi^2 - 1}} + \int_0^1 \frac{F(x)dx}{\sqrt{\xi^2 - x^2}} - \int_1^\xi \frac{dx}{\sqrt{\xi^2 - x^2}}\left[\int_0^1 \varphi(t)K(t,x)dt\right] \tag{20}$$

The stress-intensity factor (SIF) K_3 is obtained from the limiting equality

$$K_3 = \lim_{l \to 0} \sigma_{\rho\varphi}\Big|_{\rho=\rho_0}\sqrt{2l} \tag{21}$$

where l is the shortest distance from a point on the cylinder surface ($\rho = \rho_0, |z| > z_0$) to the limiting circle of the cylindrical cut. Substitution (20) into (21) and passing to the limit yield

$$K_3 = \frac{1}{2}\varphi(1)\sqrt{z_0} \tag{22}$$

If we substitute (7) into the first equation (6) and use the integral (8) value when ($\xi < 1$), we will have the difference between the displacements of the cut surfaces (crack-opening displacement). The finish result of the manipulations is the follows

$$G(u_2 - u_1) = \xi z_0 \int_\xi^1 \frac{\varphi(t)dt}{t\sqrt{t^2 - \xi^2}} \tag{23}$$

Let us consider an example. Suppose we have the boundary condition (20) in the form $\sigma_{\rho\varphi} = -\theta r_0 z_0 \xi$. Hence, the right part of the integral equation (17) is the follows $F(x) = \theta r_0 z_0 x$, where θ is an angle of twisting per unit of length. We represent this equation as

$$\frac{1}{2}\bar{\varphi}'(x) + \int_0^1 \bar{\varphi}(t)K(t,x)dt = x$$

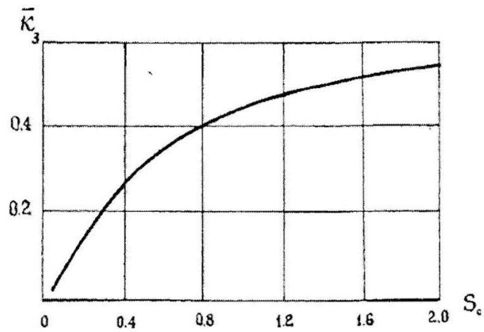

Fig. 2 Dependence of the dimensionless stress intensity factor $\overline{K}_3 = 2K_3 / \theta r_0 z^{1,5}_0$ on the ratio $S_0 = \rho_0 / z_0$

where $\overline{\varphi}(t) = \varphi(t)/\theta r_0 z_0$, kernel $K(t,x)$ is determined by formula (16). Figure 2 shows the dimensionless stress intensity factor behaviour depending on the ratio $S_0 = \rho_0 / z_0$, and $\overline{K}_3 = 2K_3 / \theta r_0 z^{1,5}_0$. Figures 3 and 4 demonstrate the dependencies of the dimensionless stress intensity factor $\overline{K}_3 = K_3 / \theta$ on the cut length and radius with different ratios S_0. It is easy to see that conditions $\dfrac{\partial K_3}{\partial z_0} > 0$, $\dfrac{\partial K_3}{\partial \rho_0} > 0$ are satisfied for all values of z_0 and ρ_0. As follows from data obtained, the area of the cut is the principal parameter for estimation of the stress intensity factor. For example, SIF takes the same value for the cuts with different geometrical parameters z_0=1, ρ_0=1.5 and z_0=1.5, ρ_0=1.

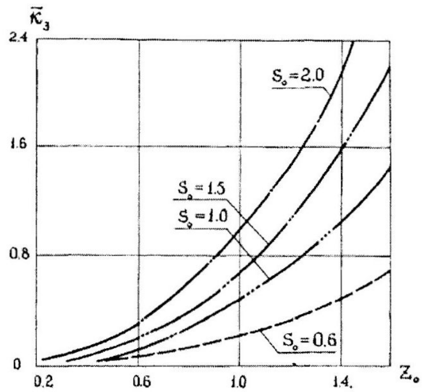

Fig.3 Dependence of the dimensionless stress intensity factor $\overline{K}_3 = K_3 / \theta$ on the cut length with different ratios $S_0 = \rho_0 / z_0$

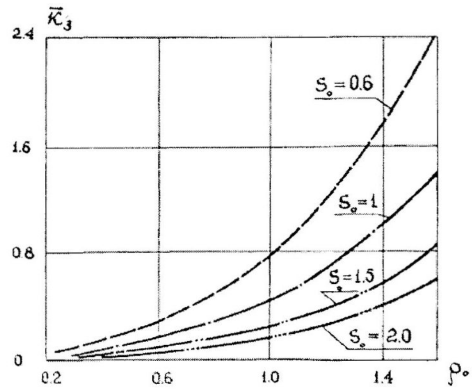

Fig.4 Dependence of the dimensionless stress intensity factor $\overline{K}_3 = K_3 / \theta$ on the cut radius with different ratios $S_0 = \rho_0 / z_0$

5. Conclusions

In this paper we obtained an analytical solution to the problem on equilibrium of elastic media with internal cylindrical crack in the field of torsional forces. The analytical solution covers all the singularities of the problem, gives a general picture of mechanical conditions of the system dependency on the changes in the problem's parameters, such as external loading, geometry of the crack, elastic constants, etc., and, thus, allows to foresee the cut's behavior when these parameters change. The results of this work show the advantages of such approach. In particular, we obtained the analytical expressions for the components of stress tensor, stress intensity factor (SIF) near the edge of cylindrical crack, the difference between the displacements of the cut surfaces (crack-opening displacement). We have found the dependencies of the dimensionless stress-intensity factor on the cylindrical crack geometry. The area of the cut surface is appeared to be the principal parameter for an estimation of the stress intensity factor.

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