ON A CONVERGENCE CLASS FOR DIRICHLET SERIES

Summary

A relation between the maximum of the modulus, the maximal term and coefficients of Dirichlet series in terms of a convergence class is investigated.

1.

Suppose that $\Lambda = (\lambda_n)$ is a sequence of positive numbers increasing to $+\infty$, and the Dirichlet series

$$F(s) = a_0 + \sum_{n=1}^{\infty} a_n \exp(s\lambda_n), \quad s = \sigma + it, \tag{1}$$

has an abscissa of absolute convergence $\sigma_a = A \in (-\infty, +\infty]$. We put $M(\sigma, F) = \sup\{|F(\sigma + it)| : t \in \mathbb{R}\}$, $\sigma < A$, and let $\mu(\sigma, F) = \max\{|a_n| \exp(\sigma\lambda_n) : n \geq 0\}$ be the maximal term and $\nu(\sigma, F) = \max\{n : |a_n| \exp(\sigma\lambda_n) = \mu(\sigma, F)\}$ be the central index of series (1).

By $\Omega(A)$ we denote the class of positive functions Φ unbounded on $(-\infty, A)$ such that the derivative Φ' is continuously differentiable, positive and increasing to $+\infty$ on $(-\infty, A)$.

The number

$$T_\Phi(F) = \varlimsup_{\sigma\uparrow A} \frac{\ln\, M(\sigma, F)}{\Phi(\sigma)}, \Phi \in \Omega(A),$$

is called Φ-*type* of F. In the case when $T_\Phi(F) = 0$ for the study of relation between the growth of $M(\sigma, F)$ and the behaviour of coefficients and exponents of series (1) we introduce a convergence Φ-class by the condition

$$\int_{\sigma_0}^{A} \frac{\Phi'(\sigma)\ln\, M(\sigma, F)}{\Phi^2(\sigma)} d\sigma < +\infty. \tag{2}$$

We establish the relation in two stages: at first we study conditions on a_n and λ_n under which

$$\int_{\sigma_0}^{A} \frac{\Phi'(\sigma)\ln\, \mu(\sigma, F)}{\Phi^2(\sigma)} d\sigma < +\infty, \tag{3}$$

and afterwards we investigate conditions under which relations (2) and (3) are equivalent. We remark that (3) implies (2) in view of Cauchy inequality $\mu(\sigma, F) \leq M(\sigma, F)$. Therefore, we need to investigate conditions under which (2) implies (3).

2.

We start from conditions under which (3) holds.

Theorem 1. *Suppose that Dirichlet series (1) has an abscissa of absolute convergence $\sigma_a = A$,*

$$\varliminf_{n\to\infty} \frac{1}{\lambda_n} \ln \frac{1}{|a_n|} = A, \quad \Phi \in \Omega(A)$$

and

$$0 < h \leq \frac{\Phi''(\sigma)\Phi(\sigma)}{(\Phi'(\sigma))^2} \leq H < +\infty, \quad \sigma \in [\sigma_0, A). \tag{4}$$

In order that (3) holds, it is necessary and in the case when

$$\varkappa_n = \frac{\ln|a_n| - \ln|a_{n+1}|}{\lambda_{n+1} - \lambda_n} \nearrow A,\ n \to \infty,$$

it is sufficient that

$$\sum_{n=n_0}^{\infty} \frac{\lambda_n - \lambda_{n-1}}{\Phi'\left(\dfrac{1}{\lambda_n} \ln \dfrac{1}{|a_n|}\right)} < +\infty. \tag{5}$$

Proof. From (4) it follows that

$$\int\limits_{\sigma_0}^{A} \frac{d\sigma}{\Phi(\sigma)} \leq \frac{1}{h} \int\limits_{\sigma_0}^{A} \frac{\Phi''(\sigma)d\sigma}{(\Phi'(\sigma))^2} = \frac{1}{h\Phi'(\sigma_0)}.$$

We put

$$B(x) = \int\limits_{x}^{A} \frac{d\sigma}{\Phi(\sigma)}.$$

Then $B(\sigma) \downarrow 0$ as $\sigma \uparrow A$.

We remark also that from (3) for every $\varepsilon > 0$ and for all $\sigma \in [\sigma_0(\varepsilon), A)$ it follows that

$$\varepsilon \geq \int\limits_{\sigma}^{A} \frac{\Phi'(t)}{\Phi^2(t)} \ln\, \mu(t, F)dt \geq \ln\, \mu(\sigma, F) \int\limits_{\sigma}^{A} \frac{\Phi'(t)}{\Phi^2(t)}dt = \frac{\ln\, \mu(\sigma, F)}{\Phi(\sigma)}. \tag{6}$$

It is well known that

$$\ln\, \mu(\sigma, F) = \ln\, \mu(\sigma_0, F) + \int\limits_{\sigma_0}^{\sigma} \lambda_{\nu(x,F)}dx, \quad \sigma_0 < \sigma < A.$$

Therefore,

$$\int\limits_{\sigma_0}^{A} \frac{\Phi'(\sigma) \ln\, \mu(\sigma, F)}{\Phi^2(\sigma)} d\sigma = \int\limits_{\sigma_0}^{A} \ln\, \mu(\sigma, F) d\left(-\frac{1}{\Phi(\sigma)}\right)$$

$$= -\left.\frac{\ln\, \mu(\sigma, F)}{\Phi(\sigma)}\right|_{\sigma_0}^{A} + \int\limits_{\sigma_0}^{A} \frac{\lambda_{\nu(\sigma,F)}d\sigma}{\Phi(\sigma)}$$

and (3) holds if and only if

$$\int\limits_{\sigma_0}^{A} \frac{\lambda_{\nu(\sigma,F)}d\sigma}{\Phi(\sigma)} < +\infty. \tag{7}$$

Let $= A0_n^0$ be the coefficients of the Newton majorant F_{mn} of series (1) and

$$\varkappa_n^0 = \frac{\ln a_n^0 - \ln a_{n+1}^0}{\lambda_{n+1} - \lambda_n}.$$

Since

$$\varliminf_{n\to\infty} \frac{1}{\lambda_n} \ln \frac{1}{|a_n|} = A,$$

then $\varkappa_n^0 \nearrow A$, $n \to \infty$, the functions F and F_{mn} have the same maximal terms $\mu(\sigma) = \mu(\sigma, F) = \mu(\sigma, F_{mn})$ and central indexes $\nu(\sigma) = \nu(\sigma, F) = \nu(\sigma, F_{mn})$ and $|a_n| \le a_n^0$ for all n. Without loss of generality we suppose that $a_0 = a_0^0 = 1$. Then

$$\varkappa_0^0 = \frac{1}{\lambda_1} \ln \frac{1}{a_1^0},$$

and for simplicity we suppose also that $\varkappa_0^0 \ge \sigma_0$.

It is known that if $\varkappa_{n-1}^0 < \varkappa_n^0$ then for all $\sigma \in [\varkappa_{n-1}^0, \varkappa_n^0)$ the equalities $\nu(\sigma) = n$ and $\mu(\sigma) = a_n \exp(\sigma\lambda_n)$ hold. Therefore,

$$\int\limits_{\sigma_0}^{A} \lambda_{\nu(\sigma,F)} \frac{d\sigma}{\Phi(\sigma)} = \sum_{n=1}^{\infty} \int_{\varkappa_{n-1}^o}^{\varkappa_n^o} \lambda_{\nu(\sigma,F)} \frac{d\sigma}{\Phi(\sigma) =} + \text{const}$$

$$= \sum_{n=1}^{\infty} \lambda_n \int_{\varkappa_{n-1}^o}^{\varkappa_n^o} \frac{d\sigma}{\Phi(\sigma)} + \text{const} = \sum_{n=1}^{\infty} \lambda_n (B(\varkappa_{n-1}^0) - B(\varkappa_n^0)) + \text{const}$$

$$= \sum_{n=1}^{\infty} (\lambda_n - \lambda_{n-1}) B(\varkappa_{n-1}^0) + \text{const}. \tag{8}$$

Since $\ln a_n^o = -\varkappa_{n-1}^0(\lambda_n - \lambda_{n-1}) - \cdots - \varkappa_0^0(\lambda_1 - \lambda_0)$, $\lambda_0 = 0$, then

$$\frac{1}{\lambda_n} \ln \frac{1}{a_n^0} = \frac{\varkappa_0^0 \lambda_1^* + \cdots + \varkappa_{n-1}^0 \lambda_n^*}{\lambda_1^* + \cdots + \lambda_n^*}, \quad \lambda_n^* = \lambda_n - \lambda_{n-1}. \tag{9}$$

Hence, firstly it follows that $\dfrac{1}{\lambda_n} \ln \dfrac{1}{a_n^0} \le \varkappa_{n-1}^o$, and in view of the decrease of B we obtain

$$\sum_{n=1}^{\infty} (\lambda_n - \lambda_{n-1}) B\left(\frac{1}{\lambda_n} \ln \frac{1}{a_n^0}\right) =\ge \sum_{n=1}^{\infty} (\lambda_n - \lambda_{n-1}) B(\varkappa_{n-1}^o). \tag{10}$$

On the other hand, if $H > 1$, $p = H/(H-1) > 1$ and $B_1(x) = B^{1/p}(x)$ then $B_1'(x) = \dfrac{1}{p} B(x)^{1/p-1} B'(x)$, and in view of (4) we have

$$B_1''(x) = \frac{1}{p} B^{1/p-2}(x) \left(B(x)B''(x) - \frac{p-1}{p} (B'(x))^2 \right)$$

$$= \frac{B^{1/p-2}(x)}{p\Phi(x)^2} \left(\Phi'(x) \int\limits_{x}^{A} \frac{d\sigma}{\Phi(= \sigma)} - \frac{1}{H} \right)$$

$$\ge \frac{B^{1/p-2}(x)}{pH\Phi(x)^2} \left(\Phi'(x) \int\limits_{x}^{A} \frac{\Phi''(\sigma) d\sigma}{(\Phi'(\sigma))^2} - 1 \right) = 0,$$

that is the function $B^{1/p}$ is convex on $[\sigma_0, A)$.

Let $p > 1$, $q = \dfrac{p}{p-1}$, $-\infty \leq a < b \leq +\infty$, f be a positive function on (a, b) such that $f^{1/p}$ is convex on (a, b), (λ_n^*) be a sequence of positive numbers,

$$(\alpha_n) \in (a, b), \qquad A_n = \frac{\lambda_1^* \alpha_1 + \cdots + \lambda_n^* \alpha_n}{\lambda_1^* + \cdots + \lambda_n^*},$$

and (μ_n) be a nonincreasing sequence of positive numbers. Then [1]

$$\sum_{n=1}^{\infty} \mu_n \lambda_n^* f(A_n) \leq q^p \sum_{n=1}^{\infty} \mu_n \lambda_n^* f(\alpha =_n).$$

If we put $f(x) = B(x)$, $\alpha_n = \varkappa_{n-1}^o$, $A_n = \dfrac{1}{\lambda_n} \ln \dfrac{1}{a_n^0}$ and $\mu_n = 1$, $n \geq 1$, then in view of (9) from here we have

$$\sum_{n=1}^{\infty} (\lambda_n - \lambda_{n-1}) B\left(\frac{1}{\lambda_n} \ln \frac{1}{a_n^0}\right) \leq q^p \sum_{n=1}^{\infty} (\lambda_n - \lambda_{n-1}) B(\varkappa_{n-1}^o). \tag{11}$$

From (8), (10) and (11) it follows that (7) and well then (3) holds if and only if

$$\sum_{n=1}^{\infty} (\lambda_n - \lambda_{n-1}) B\left(\frac{1}{\lambda_n} \ln \frac{1}{a_n^0}\right) < +\infty. \tag{12}$$

Since $|a_n| \leq a_n^0$ for each Dirichlet series (1), $|a_n| = a_n^0$ provided $\varkappa_n \nearrow A$, $n \to \infty$ and (4) implies

$$\frac{1}{H\Phi'(x)} \leq B(x) \leq \frac{1}{h\Phi'(x)},$$

we obtain from (12) the conclusion of Theorem 1.

3.

In order to establish conditions under which (3) implies (2) one can use different methods. We will dwell on two such methods. One of them is based on the following lemma.

Lemma 1. *Suppose that Dirichlet series* (1) *has an abscissa of absolute convergence* $\sigma_a = A$ *and function* f *is continuous and increasing to* A *on* $(-\infty, A)$. *For* $\sigma < A$ *we put*

$$p(\sigma) = \sup\left\{\frac{\sigma - t}{f(\sigma) - f(t)} : \sigma_0 \leq t < \sigma\right\},$$

and let g be a function continuous on $(-\infty, +\infty)$ such that $g(x) = f^{-1}(x)$ on $(-\infty, A)$ and if $A < +\infty$ then $g(x) = A$ for $x \geq A$.

If

$$\sum_{n=1}^{\infty} |a_n| \exp\left\{\lambda_n g\left(\frac{1}{\lambda_n} \ln \frac{1}{|a_n|}\right)\right\} \leq K_0 < +\infty, \tag{13}$$

then for all $\sigma < A$

$$M(\sigma, F) \leq K_0(\mu(f(\sigma), F))^{p(\sigma)} + K_0 + |a_0|. \tag{14}$$

Lemma 1 is proved in [2] for entire Dirichlet series (i. e. $A = +\infty$) and in [3] for the case $-\infty < A \leq +\infty$.

For $\Phi \in \Omega(A)$ let φ be an inverse function to Φ' and

$$\Psi(x) = x - \frac{\Phi(x)}{\Phi'(x)}$$

be a function associated with Φ in the sense of Newton. Then φ is continuously differentiable and increasing to A on $(0, +\infty)$ and Ψ is continuously differentiable and increasing to A on $(-\infty, A)$.

Theorem 2. *Let $A \in (-\infty, +\infty]$, $\Phi \in \Omega(A)$, $\Phi(\sigma) = O(\Phi(\Psi(\sigma))), \sigma \uparrow A$ and (4) holds. Suppose that Dirichlet series (1) has the abscissa of absolute convergence*

$$\sigma_a = A, \quad \frac{1}{\lambda_n} \ln \frac{1}{|a_n|} < A,\ n \geq n_0,$$

and

$$\sum_{n=1}^{\infty} |a_n| \exp\left\{\lambda_n \Psi\left(\frac{1}{\lambda_n} \ln \frac{1}{|a_n|}\right)\right\} < +\infty. \tag{15}$$

Then the relations (2) and (3) are equivalent.

Proof. Since

$$\Psi'(\sigma) = \frac{\Phi''(\sigma)\Phi(\sigma)}{(\Phi'(\sigma))^2},$$

from (4) we have $0 < h \leq \Psi'(\sigma) \leq H < +\infty$, $\sigma \in [\sigma_0, A)$. We choose in Lemma 1 $f(\sigma) = \Psi^{-1}(\sigma)$. Then $g(\sigma) = \Psi(\sigma)$ and

$$\begin{aligned} p(\sigma) &= \sup\left\{\frac{\Psi(\sigma) - \Psi(t)}{\sigma - t} : \sigma_0 \leq t < \sigma\right\} \\ &= \sup\{\Psi'(\xi) : \sigma_0 \leq t < \xi < \sigma\} < H. \end{aligned} \tag{16}$$

Since (15) implies (13), by Lemma 1 we have (14), i. e. in view of (16) $\ln M(\sigma, F) \le H \ln \mu(\Psi^{-1}(\sigma), F) + O(1)$, $\sigma \uparrow A$. Therefore, if (3) holds then

$$\int_{\sigma_0}^{A} \frac{\Phi'(\sigma) \ln M(\sigma, F)}{\Phi^2(\sigma)} d\sigma \le H \int_{\sigma_0}^{A} \frac{\Phi'(\sigma) \ln \mu(\Psi^{-1}(\sigma), F)}{\Phi^2(\sigma)} d = \sigma + \text{const}$$

$$\le H \int_{\sigma_0}^{A} \frac{\Phi^2(\Psi^{-1}(\sigma))\Psi'(\Psi^{-1}(\sigma))}{\Phi^2(\sigma)} \frac{\Phi'(\Psi^{-1}(\sigma)) \ln \mu(\Psi^{-1}(\sigma), F)}{\Phi^2(\Psi^{-1}(\sigma))} d\Psi^{-1}(\sigma) + \text{const}$$

$$\le H_1 \int_{\sigma_0}^{A} \frac{\Phi'(\Psi^{-1}(\sigma)) \ln \mu(\Psi^{-1}(\sigma), F)}{\Phi^2(\Psi^{-1}(\sigma))} d\Psi^{-1}(\sigma) + \text{const} < +\infty,$$

because $\Psi'(\sigma) = O(1)$ and $\Phi(\Psi^{-1}(\sigma)) = O(\Phi(\sigma))$ as $\sigma \uparrow A$. Theorem 2 is proved.

Another metods is based on the following lemma.

Lemma 2 [4]. *Suppose that Dirichlet series (1) has an abscissa of absolute convergence $\sigma_a = A \in (-\infty, +\infty]$ and $\Phi \in \Omega(A)$. In order that $\ln \mu(\sigma, F) < \Phi(\sigma)$ for all $\sigma_0 \le \sigma < A$, it is necessary and sufficient that $\ln |a_n| < -\lambda_n \Psi(\varphi(\lambda_n))$ for all $n \ge n_0$.*

Theorem 3. *Let $A \in (-\infty, +\infty]$ and $\Phi \in \Omega(A)$. Suppose that Dirichlet series (1) has an abscissa of absolute convergence $\sigma_a = A$ and there exists a positive increasing to $+\infty$ function γ on $(-\infty, A)$ such that*

$$\int_{\sigma_0}^{A} \frac{\Phi'(\sigma) \ln n(\gamma(\sigma))}{\Phi^2(\sigma)} d\sigma < +\infty, \quad n(t) = \sum_{\lambda_n \le t} 1, \tag{17}$$

and

$$\sum_{n=1}^{\infty} \exp\{-\lambda_n \Psi(\varphi(\lambda_n)) + \lambda_n \gamma^{-1}(\lambda_n)\} < +\infty. \tag{18}$$

Then the relations (2) and (3) are equivalent.

Proof. From (6) it follows that $\ln \mu(\sigma, F) < \Phi(\sigma)$ for all $\sigma_0 \le \sigma < A$ and by Lemma 2 $\ln |a_n| < -\lambda_n \Psi(\varphi(\lambda_n))$ for all $n \ge n_0$. Therefore,

$$\begin{aligned} M(\sigma, F) &\le \sum_{\lambda_n < \gamma(\sigma)} |a_n| \exp\{\lambda_n \sigma\} + \sum_{\lambda_n \ge \gamma(\sigma)} |a_n| \exp\{\lambda_n \sigma\} \\ &\le \mu(\sigma, F) n(\gamma(\sigma)) + \sum_{\lambda_n \ge \gamma(\sigma)} \exp\{-\lambda_n \Psi(\varphi(\lambda_n)) + \lambda_n \sigma\} \\ &\le \mu(\sigma, F) n(\gamma(\sigma)) + \sum_{\lambda_n > \gamma(\sigma)} \exp\{-\lambda_n \Psi(\varphi(\lambda_n)) + \lambda_n \gamma^{-1}(\lambda_n)\}, \end{aligned}$$