Generalized separation of variables for nonlinear equation $u_{tt} = F(u)u_{xx} + aF'(u)u_x^2$

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Abstract

We consider the equation $u_{tt} = F(u)u_{xx} + aF'(u)u_x^2$, where $F(u)$ and $a \neq 0$ are an arbitrary function and constant, correspondingly. The problem is studied for which functions $F(u)$ it admits ansätz $t = w_1(x)d(u) + w_2(x)$, which reduces this equation to a system of two ordinary differential equations with unknown functions $w_1(x)$ and $w_2(x)$. For these equations classes of exact solutions with generalized separation of variables are constructed, which can not be obtained by the method of classical group analysis.

Key words: nonlinear hyperbolic equations; generalized separation of variables

1 Introduction

In recent decades, much attention is paid to search exact solutions of nonlinear equations of mathematical physics with separated variables. At that as the concept of separation variables was generalized and the range of equations under studying was expanded. So in the papers [1,2] some types of parabolic and hyperbolic equations with quadratic nonlinearity are described that admit exact solutions with generalized separation of variables of the form $u = \varphi(x)\psi(t) + \chi(t)$. In the paper [3] it is described all nonlinear equations of theory of waves and heat conduction of the form

$$u_{tt} + u_{xx} = f(u)$$

that admit exact solutions with functional separation of variables of the form

$$u(x,t) = F(z), \qquad z = \varphi(x) + \psi(t).$$

Using generalized conditional-group approach to functional separation of variables in [4] new non-Lie solutions of the nonlinear wave equation is obtained.

There is a connection between generalized separation of variables and nonclassical symmetries (Lie–Bäcklund symmetries). In particularly the papers [5,6] are devoted this problem, where principally new separable solutions of some nonlinear evolution equations are obtained.

Study presented in this paper are related to the range of equations considered above. Here the nonlinear equation

$$\frac{\partial^2 u}{\partial t^2} = F(u)\frac{\partial^2 u}{\partial x^2} + aF'(u)\left(\frac{\partial u}{\partial x}\right)^2 \tag{1}$$

is considered that found in problems of wave and gas dynamics, in the theory of liquid crystals and has a number of other applications. It is known that in general case this equation has exact solutions of the form

$$u(x,t) = \omega(z), \ z = kx + \lambda t,$$

$$u(x,t) = \omega(\xi), \ \xi = \frac{x+b}{t+c},$$

where k, λ, b, c are an arbitrary constants.

In the case $a = 1$ group properties of equations (1) are studied by the method of S. Lie in [7].

If $F(u) = \lambda u^k$ then equation (1) has an exact solution as a product of functions of different arguments

$$u = \varphi(x)\psi(t),$$

and if $F(u) = \lambda \exp(bu)$, then (1) has an exact solution as a sum of functions of different arguments:

$$u = \varphi(x) + \psi(t).$$

Qualitative analysis of a structure of solutions of equation (1) is studied in the papers [8,9].

The important case for equation (1) with $F(u) = \lambda u$ is considered in [10–12]. In the papers [10, 11] for given equation the method of construction exact solutions of the form

$$u(x,t) = \sum_{i=1}^{k} f_i(t) a_i(x) \tag{2}$$

is used, which based on finding finite dimensional subspaces invariant under a differential operator that corresponds to the right side of equation (1). Solutions of form (2) for $k > 1$ are called solutions of generalized separation of variables. For this case in the paper [12] the ansätz

$$u = \sum_{i=1}^{m} w_i(t) a_i(x) + f(x,t), \ m \geq 1. \tag{3}$$

has been applied for obtaining exact solutions of form (2). Ansätz (3) contains unknown function $f(x,t)$, m unknown functions $a_i(x)$ and m unknown functions $w_i(t)$ which specified under the condition that ansätz (3) reduces equation (1) to a system of m ordinary differential equations with unknown functions $w_i(t)$. If in ansätz (3) $m = 1$ then this system is reduced to an ordinary differential equation with unknown function $w_1(t)$.

If in a given differential equation we perform the substitution $t = t(u,x)$ then we have a differential equation with the independent variables u, x and the dependent variable t. For construction solutions with the generalized separation of variables of such equation we can use the ansätz

$$t = \sum_{i=1}^{m} w_i(x) a_i(u) + f(u,x), \tag{4}$$

which is obtained from ansätz (3) as a result of performing of the substitution $u \mapsto t$, $x \mapsto u$, $t \mapsto x$.

In this paper the question is investigated for which functions $F(u)$ equation (1) admits ansatz (4). Using ansatzes (3) and (4) for these equations classes of exact solutions with the generalized separation of variables are constructed which can not be obtained by means of the classic group analysis method.

2 Equations (1) that admit ansatzes $t = w_1(x)d(u) + w_2(x)$

Let us find out for which functions $F(u)$ equation (1) admits the following ansatzes

$$t = w_1(x)d(u) + w_2(x). \tag{5}$$

We define the functions $w_1(x)$, $w_2(x)$ and $d(u)$ in (5) up to transformations

$$d = \lambda_1 \hat{d} + \lambda_2, \ w_1 = \lambda_1^{-1}\hat{w}_1, \ w_2 = -\lambda_1^{-1}\lambda_2\hat{w}_1 + \hat{w}_2, \tag{6}$$

$\hat{d} = \hat{d}(u)$, $\hat{w}_1 = \hat{w}_1(x)$, $\hat{w}_2 = \hat{w}_2(x)$, $\lambda_1, \lambda_2 \in \mathbb{R}$, $\lambda_1 \neq 0$. Transformations (6) reduce ansatze (5) to ansatze

$$t = \hat{w}_1(x)\hat{d}(u) + \hat{w}_2(x),$$

without changing its structure, but making it possible considerably to simplify it by choosing the function $d(u)$.

Substituting (5) in (1) we have the relation

$$\begin{aligned} &Fd(d')^2w_1w_1'' - [2Fd(d')^2 + aF'd^2d' - Fd^2d''](w_1')^2 + F(d')^2w_1w_2'' - \\ &-[2F(d')^2 + 2aF'dd' - 2Fdd'']w_1'w_2' - [-Fd'' + aF'd'](w_2')^2 - d'' = 0. \end{aligned} \tag{7}$$

The system of two ordinary differential equations with unknown functions w_1 and w_2 can be received in following way. In equation (7) the functions $Fd(d')^2$, $F(d')^2$ under w_1w_1'' and w_1w_2'' are linear independent over $\mathbb{R}$. Impose the condition on the coefficients under the functions $(w_1')^2$, $w_1'w_2'$, $(w_2')^2$, and also on the function d'', so that they can be represented as a linear combination over $\mathbb{R}$ of the functions $Fd(d')^2$, $F(d')^2$. Then we have

$$\begin{aligned} &aF'd^2d' - Fd^2d'' = \beta_1 Fd(d')^2 + \gamma_1 F(d')^2, \\ &aF'dd' - Fdd'' = \beta_2 Fd(d')^2 + \gamma_2 F(d')^2, \\ &aF'd' - Fd'' = \beta_3 Fd(d')^2 + \gamma_3 F(d')^2, \\ &d'' = \beta Fd(d')^2 + \gamma F(d')^2, \end{aligned} \tag{8}$$

β, β_i, γ, $\gamma_i \in \mathbb{R}$ $(i = 1, 2, 3)$. From the first three equations of system (8) it follows that $\beta_i = \gamma_i = 0$ for $i = 1, 2, 3$. Therefore this system is equivalent to the system of two equations

$$aF'd' - Fd'' = 0, \tag{9}$$

$$d'' = \beta F d (d')^2 + \gamma F (d')^2. \tag{10}$$

We find from equations (9) that

$$F = \lambda (d')^{1/a}, \tag{11}$$

where $\lambda \neq 0$ is a constant, and taking into account (10) we have that

$$d'' = \lambda(\beta d + \gamma)(d')^{(1+2a)/a}. \tag{12}$$

So, if equation (1) admits ansatzes (5), then the function $F(u)$ is defined by formula (11), where $d = d(u)$ is an arbitrary solution of ordinary differential equation (12).

To construct solutions of equation (1) it is necessary to integrate equation (12) and the corresponding system of ordinary differential equations with unknown functions w_1, w_2, which is given below. Let us consider the case $\beta = 0$.

1) If $a \neq -1$, then performing the appropriate transformation (6), we can assume that in equation (12) $\lambda\gamma = \varepsilon \dfrac{a}{a+1}$, $\varepsilon = \pm 1$, that is

$$d'' = \varepsilon \frac{a}{a+1} (d')^{(1+2a)/a}, \ a \neq -1. \tag{13}$$

The solution of (13) is the function

$$d = -\varepsilon(1+a)(-\varepsilon u + b)^{1/(1+a)} + c, \tag{14}$$

here b, c are arbitrary constants. Taking into account transformation (6) we can assume that $c = 0$ in (14). From (11) and (14) we obtain

$$F(u) = \lambda(-\varepsilon u + b)^{-1/(1+a)}, \ a \neq -1. \tag{15}$$

Performing in (14) the change $-\varepsilon u + b = U$, we find that

$$F = \lambda U^{-1/(1+a)}, \tag{16}$$

and equation (1) to determine the function $U = U(x,t)$ takes the form

$$\frac{\partial^2 U}{\partial t^2} = \lambda U^{-1/(1+a)} \frac{\partial^2 U}{\partial x^2} - \frac{\lambda a}{1+a} U^{-(2+a)/(1+a)} \left(\frac{\partial U}{\partial x} \right)^2. \tag{17}$$

Thus, the problem of construction of solutions is reduced to integration of equation (17). Substituting (11) into equation (7), and taking into account (9) we obtain the equation for determination the functions w_1 and w_2:

$$d(d')^2 [w_1 w_1'' - 2(w_1')^2] + (d')^2 \left[w_1 w_2'' - 2 w_1' w_2' - \frac{\varepsilon a}{\lambda(a+1)} \right] = 0.$$

It splits into the system of two equations

$$w_1 w_1'' - 2(w_1')^2 = 0, \quad w_1 w_2'' - 2 w_1' w_2' - \frac{\varepsilon a}{\lambda(a+1)} = 0.$$

So, in the case when the function F is determined by formula (16), equation (1) (i.e. equation (17)) has the solution

$$U^{1/(1+a)} = \varphi_1(x)t + \varphi_2(x),$$

where the functions

$$\varphi_1(x) = -\frac{\varepsilon}{a+1}w_1^{-1}, \quad \varphi_2(x) = \frac{\varepsilon}{a+1}w_1^{-1}w_2$$

are fount from the system of ordinary differential equations

$$\varphi_1'' = 0, \quad \varphi_2'' = \frac{a}{\lambda}\varphi_1^2.$$

Let us give another exact solutions of equation (17). After completing the replacement

$$U = V^{1+a}, \quad x = \tau, \quad t = \xi, \tag{18}$$

we reduce equation (17) to the form

$$\frac{\partial^2 V}{\partial \tau^2} = \frac{1}{\lambda}V\frac{\partial^2 V}{\partial \xi^2} + \frac{a}{\lambda}\left(\frac{\partial V}{\partial \xi}\right)^2. \tag{19}$$

In the papers [10, 11] solutions of equation (19) are searched in the form of finite sum

$$V = \sum_{i=1}^{k} f_i(\tau)a_i(\xi), \tag{20}$$

where the system of functions $a_i(\xi)$ is a basis of subspace invariant under an operator that corresponds to the right side of equation (19). It means that the system of functions $a_i(\xi)$ in formula (20) is looked for as a solution of a nonlinear system of k ordinary differential equations. There is no effective methods of finding partial solutions, and the more of finding of a general solution of such system of equation, even in the case $k = 2$. To eliminate this problem the system of coordinate functions $a_i(\xi)$ in formula (20) is specified a priori, and the corresponding functions $f_i(\tau)$ are determined by the method of undetermined coefficients. As result the following classis of exact solutions of equation (19) are constructed [10, 11]:

$$V = \sum_{i=0}^{2} \mu_i(\tau)\xi^i,$$

$$V = \sum_{i=0}^{3} \mu_i(\tau)\xi^i, \text{ if } \ a = -\frac{2}{3},$$

$$V = \sum_{i=0}^{4} \mu_i(\tau)\xi^i, \text{ if } \ a = -\frac{3}{4}.$$

In the paper [12] for finding exact solutions of equation (19) it is used the ansatz of form (3):

$$V = w(\tau)d(\xi) + f(\tau, \xi),$$

where the functions $d(\xi)$, $f(\tau,\xi)$ and $w(\tau)$ are defined under condition that the ansatz reduces (19) to ordinary differential equations with unknown function $w(\tau)$. As result the following classis of exact solutions of equation (19) are constructed [12] (give some of them):

$$V = 6\lambda\tau^{-2}\xi^2\ln|\xi| + \left(c_1\tau^3 + c_2\tau^{-2} + \frac{18\lambda}{5}\tau^{-2}\ln|\tau|\right)\xi^2, \text{ if } a = -\frac{1}{2};$$

$$V = \frac{3\lambda}{1+2a}\lambda\tau^{-2}\xi^2 + \tau^{1/2}\xi^\alpha[c_1\cos(\sigma\ln\tau) + c_2\sin(\sigma\ln\tau)],$$

if $\sigma^2 = 3\alpha^2 - 9\alpha - \frac{1}{4}$, where $\alpha = \frac{1}{1+a}$, $a \neq -1, -\frac{1}{2}, -\frac{2}{3}$;

$$V = \frac{3\lambda}{1+2a}\lambda\tau^{-2}\xi^2 + \tau^{1/2}\xi^\alpha[c_1\tau^\sigma + c_2\tau^{-\sigma}],$$

if $\sigma^2 = \frac{1}{4} - 3\alpha^2 + 9\alpha > 0$ where $\alpha = \frac{1}{1+a}$, $a \neq -1, -\frac{1}{2}, -\frac{2}{3}$;

$$V = \frac{3\lambda}{1+2a}\lambda\tau^{-2}\xi^2 + \tau^{1/2}\xi^\alpha[c_1 + c_2\ln\tau],$$

if $\alpha = \frac{9 \pm 2\sqrt{21}}{6}$, where c_1, c_2 are an arbitrary constants. All solutions of equation (19), considered above can not be constructed by the classical symmetry analysis approach.

Using solutions of equation (19) given in [10–12] and making in them the change of variables V, τ, ξ, we obtained the solutions of equation (17) with the help of formula (18). For instance, consider such exact solution of equation (19):

$$V = \frac{3\lambda}{1+2a}\lambda\tau^{-2}\xi^2 + \tau^{1/2}\xi^\alpha[c_1\cos(\sigma\ln\tau) + c_2\sin(\sigma\ln\tau)],$$

if $\sigma^2 = 3\alpha^2 - 9\alpha - \frac{1}{4}$, where $\alpha = \frac{1}{1+a}$, $a \neq -1, -\frac{1}{2}, -\frac{2}{3}$, c_1, c_2 are an arbitrary functions. Returning to the variables U, x, t with the help of (18) we has the exact solution of equation (17):

$$U^{1/1+a} = \frac{3\lambda}{1+2a}x^{-2}t^2 + x^{1/2}t^\alpha[c_1\cos(\sigma\ln x) + c_2\sin(\sigma\ln x)].$$

2) If $a = -1$, then equation (12) has a view

$$d'' = bd', \quad b = \lambda\gamma. \tag{21}$$

The solution of equation (21) is the function

$$d = \frac{1}{b}\exp(bu) + c, \tag{22}$$

where c is a constant. Performing appropriate transformation (6) we can assume that $c = 0$ in (22), and consequently taking into account (11)

$$F(u) = \lambda(d')^{-1} = \lambda\exp(-bu).$$

In this case equation (1) has the view

$$\frac{\partial^2 u}{\partial t^2} = \lambda \exp(-bu)\frac{\partial^2 u}{\partial x^2} + \lambda b \exp(-bu)\left(\frac{\partial u}{\partial x}\right)^2. \tag{23}$$

Using (7) we get the system of differential equations for finding of the functions w_1 and w_2:

$$w_1 w_1'' - 2(w_1')^2 = 0, \quad w_1 w_2'' - 2w_1' w_2' - \frac{b}{\lambda} = 0.$$

Integrating the system and taking into account ansatz (5) we get the exact solution of equation (23):

$$u = \frac{1}{b}\ln\left[bc_1(x+c_2)t - \frac{b^2c_1^2}{12\lambda}(x+c_2)^4 + c_3(x+c_2) + c_4\right],$$

here c_1, c_2, c_3, c_4 are arbitrary constants.

3 Exact solutions of the form $t = w_1(x)d(u) + w_2(x)$ of equation (1)

Let in equation (12) $\beta \neq 0$. If $a \neq -\frac{1}{2}$, then performing the appropriate transformation (6), we can assume that in equation (12) $\lambda\beta = \frac{2a\varepsilon}{(1+a)^2}$, $\varepsilon = \pm 1$ $\gamma = 0$. Let us consider the case when $\varepsilon = -1$, notable:

$$d'' = -\frac{2a}{(1+2a)^2} d(d')^{(1+2a)/a}, \quad a \neq -\frac{1}{2}. \tag{24}$$

A partial solution of equation (24) is the function

$$d = \varepsilon(1+2a)(\varepsilon u + b)^{1/(1+2a)}, \quad \varepsilon = \pm 1, \tag{25}$$

and so

$$F(u) = \lambda(\varepsilon u + b)^{-2/(1+2a)}, \quad a \neq -\frac{1}{2}. \tag{26}$$

Equation (1) which corresponds to the function (26) has the form

$$\frac{\partial^2 u}{\partial t^2} = \lambda(\varepsilon u + b)^{-2/(1+2a)}\frac{\partial^2 u}{\partial x^2} - \frac{2\lambda\varepsilon a}{(1+2a)}(\varepsilon u + b)^{-(3+2a)/(1+2a)}\left(\frac{\partial u}{\partial x}\right)^2, \tag{27}$$

and is reduced to equation

$$\frac{\partial^2 U}{\partial t^2} = \lambda U^{-2/(1+2a)}\frac{\partial^2 U}{\partial x^2} - \frac{2\lambda a}{1+2a} U^{-(3+2a)/(1+2a)}\left(\frac{\partial U}{\partial x}\right)^2 \tag{28}$$

by means of the change $\varepsilon u + b = U$. Thereby if equation (1) admits ansatz (5), where the function $d = d(u)$ is determined by formula (25), then the equation is local equivalent of equation (28).

For construction of solutions of equation (28) it is enough to define the functions $w_1(x)$ and $w_2(x)$ in ansatz (5). The system of equations for determination of these functions has the form

$$w_1 w_1'' - 2(w_1')^2 + \frac{2a}{\lambda(1+2a)^2} = 0, \; w_1 w_2'' - 2w_1' w_2' = 0 \tag{29}$$

and is obtained from equation (7) and formulae (25), (26). Taking into account (5) equation (28) has the solution

$$U^{1/1+2a} = \varphi_1(x)t + \varphi_2(x),$$

where the functions

$$\varphi_1(x) = \frac{1}{1+2a} w_1^{-1}, \; \varphi_2(x) = -\frac{1}{1+2a} w_1^{-1} w_2$$

are determined from the system of ordinary differential equations

$$\varphi_1'' = \frac{2a}{\lambda}\varphi_1^3, \quad \varphi_2'' = \frac{2a}{\lambda}\varphi_1^2\varphi_2. \tag{30}$$

The first equation (30) is solved independently and its solutions are expressed in terms of Jacobi elliptic functions. Infinite series of exact solutions of this equation are given in [13]. The second equation (30) has the partial solution $\varphi_2 = \varphi_1$, then its general solution can be written as (see [14])

$$\varphi_2 = c_1\varphi_1 + c_2\varphi_1 \int \frac{dx}{\varphi_1^2},$$

here c_1, c_2 are an arbitrary constants.

Note that by the changing of variable $U = V^{1+2a}$ equation (28)is reduced to the equation

$$\frac{\partial^2 V}{\partial x^2} = \frac{1}{\lambda} V^2 \frac{\partial^2 V}{\partial t^2} + \frac{2a}{\lambda} V \left(\frac{\partial V}{\partial t}\right)^2.$$

The given equation for $a = -\dfrac{1}{4}$ has the following solution

$$V = \mu_2(x)t^2 + \mu_1(x)t + \mu_0(x),$$

where functions $\mu_i(x)$ are determined from the system of ordinary differential equations

$$\mu_2'' = \mu_2\left(-\frac{1}{2\lambda}\mu_1^2 + \frac{2}{\lambda}\mu_0\mu_2\right),$$

$$\mu_1'' = \mu_1\left(-\frac{1}{2\lambda}\mu_1^2 + \frac{2}{\lambda}\mu_0\mu_2\right),$$

$$\mu_0'' = \mu_0 \left(-\frac{1}{2\lambda}\mu_1^2 + \frac{2}{\lambda}\mu_0\mu_2 \right).$$

So, equation (28) for $a = -\dfrac{1}{4}$ has the solution

$$U = V^{1/2} = (\mu_2(x)t^2 + \mu_1(x)t + \mu_0(x))^{1/2}.$$

Note also that the equation

$$\frac{\partial^2 U}{\partial t^2} = \lambda U^{-k/(1+ka)} \frac{\partial^2 U}{\partial x^2} - \frac{k\lambda a}{1+ka} U^{-[k(a+1)+1]/(1+ka)} \left(\frac{\partial U}{\partial x} \right)^2,$$

which is the generalization of equations (17) and (28), is reduced to the equation

$$\frac{\partial^2 V}{\partial t^2} = \frac{1}{\lambda} V^k \frac{\partial^2 V}{\partial x^2} + \frac{ka}{\lambda} V^{k-1} \left(\frac{\partial V}{\partial x} \right)^2$$

by the changing of variable $U = V^{1+ka}$. Consider the cases when equation (12) with $\beta \neq 0$ can be completely integrated. We can assume that in equation (12) $\gamma = 0$.

a) The case $a = -1/2$. Equation (12) takes the form

$$d'' = \lambda\beta d. \tag{31}$$

Taking into account that the function $d = d(u)$ in ansatz (5) is a solution of equation (31) and determined up to the transformation $d \to \lambda_1 d + \lambda_2$, we can assume that

$$d = \begin{cases} \gamma^{-1}\sinh(\gamma u + b), & \text{if } \lambda\beta = \gamma^2, \\ \gamma^{-1}\sin(\gamma u + b), & \text{if } \lambda\beta = -\gamma^2. \end{cases} \tag{32}$$

From (11) we have

$$F(u) = \begin{cases} \lambda\cosh^{-2}(\gamma u + b), & \text{if } \lambda\beta = \gamma^2, \\ \lambda\cos^{-2}(\gamma u + b), & \text{if } \lambda\beta = -\gamma^2. \end{cases} \tag{33}$$

Performing in (33) the substitution $\gamma u + b = U$, we obtain

$$F = \lambda\cosh^{-2} U \text{ or } F = \lambda\cos^{-2} U. \tag{34}$$

Equations (1), that correspond of functions (34), have the form

$$\frac{\partial^2 U}{\partial t^2} = \lambda\cosh^{-2} U \frac{\partial^2 U}{\partial x^2} + \lambda\cosh^{-2} U \tanh U \left(\frac{\partial U}{\partial x} \right)^2, \tag{35}$$

$$\frac{\partial^2 U}{\partial t^2} = \lambda\cos^{-2} U \frac{\partial^2 U}{\partial x^2} - \lambda\cos^{-2} U \tan U \left(\frac{\partial U}{\partial x} \right)^2. \tag{36}$$

So, if equation (1) admits ansatz (5), where the function $d = d(u)$ is a solution of equation (31), then equation (1) is locally equivalent to one of equations (35) or (36).

Let us construct solutions of equations (35) and (36). The system of equations for finding of the functions w_1 and w_2 has a view

$$w_1 w_1'' - 2(w_1')^2 + \frac{\varepsilon\gamma^2}{\lambda} = 0, \quad w_1 w_2'' - 2w_1' w_2' = 0,$$

where $\varepsilon = -1$ for equation (35) and $\varepsilon = 1$ for equation (36), and is obtained from equation (7) and formulae (32), (33). On the ground (5) equations (35) and (36) have correspondingly the following solutions

$$U = \mathrm{Arsh}(\varphi_1(x)t + \varphi_2(x)), \tag{37}$$

$$U = \arcsin(\varphi_1(x)t + \varphi_2(x)), \tag{38}$$

where the functions $\varphi_1(x) = w_1^{-1}$, $\varphi_2(x) = -w_1^{-1}w_2$ are determined from from the system of ordinary differential equations

$$\varphi_1'' = \frac{\varepsilon}{\lambda}\varphi_1^3, \qquad \varphi_2'' = \frac{\varepsilon}{\lambda}\varphi_1^2\varphi_2.$$

In the cases $F = \lambda \cosh^{-2}(\gamma u + b)$ and $F = \lambda \cos^{-2}(\gamma u + b)$ the finding of solutions of corresponding equations (1) is reduced to the applying of the change $U = \gamma u + b$ to solutions (37) and (38).

b) The case $a = -1$. Equation (12) has the view

$$d'' = \lambda\beta d d'. \tag{39}$$

Performing the corresponding transformation $d \to \lambda_1 d + \lambda_2$, we can assume that in equation (39) $\lambda\beta = 2$. Integrating equation (39), we find the following solutions: $d = b\tan(bu + \gamma_0)$ and $d = -b\tanh(bu + \gamma_0)$. Since in ansatz (5) the function $d = d(u)$ is defined up to the transformation $d \to \lambda_1 d + \lambda_2$, $\lambda_1 \in \mathbb{R}$, we can assume that in ansatz (5)

$$d = b^{-1}\tan(bu + \gamma_0) \quad \text{or} \quad d = -b^{-1}\tanh(bu + \gamma_0), \tag{40}$$

where b, γ_0 are arbitrary constants. So, taking to account (11) we have

$$F(u) = \lambda\cos^2(bu + \gamma_0) \quad \text{or} \quad F(u) = -\lambda\cosh^2(bu + \gamma_0). \tag{41}$$

Making the change of variable $bu + \gamma_0 = U$ in (41), we obtain

$$F = \lambda\cos^2 U \quad \text{or} \quad F = -\lambda\cosh^2 U \tag{42}$$

and equations (1) with function (42):

$$\frac{\partial^2 U}{\partial t^2} = \lambda\cos^2 U \frac{\partial^2 U}{\partial x^2} + 2\lambda\cos^2 U \tan U \left(\frac{\partial U}{\partial x}\right)^2, \tag{43}$$

$$\frac{\partial^2 U}{\partial t^2} = -\lambda\cosh^2 U \frac{\partial^2 U}{\partial x^2} + 2\lambda\cosh^2 U \tanh U \left(\frac{\partial U}{\partial x}\right)^2. \tag{44}$$

Thus, if equation (1) admits ansatz (5), where the function $d = d(u)$ is a solution of equation (39), then equation (1) is locally equivalent to equation (43), or equation (44).

To construct solutions of equations (43) and (44) it is sufficient to determine the functions $w_1(x)$ and $w_2(x)$. The system of equations for determining of the functions w_1 and w_2 takes the form

$$w_1 w_1'' - 2(w_1')^2 - \frac{2b^2}{2\lambda} = 0, \quad w_1 w_2'' - 2w_1' w_2' = 0$$

and is obtained from equation (7) and formulae (40), (41). On the basis of (5) equations (43) and (44) have, respectively, such solutions:

$$U = \arctan(\varphi_1(x)t + \varphi_2(x)), \tag{45}$$

$$U = \operatorname{Arth}(\varphi_1(x)t + \varphi_2(x)), \tag{46}$$

where the fuctions $\varphi_1(x) = bw_1^{-1}$, $\varphi_2(x) = -bw_1^{-1}w_2$ are determined from the system of ordinary differential equations

$$\varphi_1'' = -\frac{2}{\lambda}\varphi_1^3, \quad \varphi_2'' = -\frac{2}{\lambda}\varphi_1^2\varphi_2.$$

c) The case $a = 1$. We can assume that equation (12) has the form

$$d'' = -6d(d')^3. \tag{47}$$

The solution of equation (47) is a function $d = d(u)$, which is determined from the equation

$$d^3 + pd + q = u, \tag{48}$$

where p, q are arbitrary constants. Then we have

$$d' = \frac{1}{3d^2 + p}.$$

So, from (11) it follows

$$F(u) = \frac{\lambda}{3d^2 + p}\ . \tag{49}$$

Equation (1) with function (49) has the form

$$\frac{\partial^2 u}{\partial t^2} = \frac{\lambda}{3d^2 + p}\frac{\partial^2 u}{\partial x^2} - \frac{6\lambda d}{(3d^2 + p)^3}\left(\frac{\partial u}{\partial x}\right)^2. \tag{50}$$

To determine the functions w_1 and w_2 we obtain the system of equations

$$w_1 w_1'' - 2(w_1')^2 + \frac{6}{\lambda} = 0, \quad w_1 w_2'' - 2w_1' w_2' = 0.$$

Taking into account (5) we have the following solution for equation (50):

$$d(u) = \varphi_1(x)t + \varphi_2(x),$$

here the functions $\varphi_1(x) = w_1^{-1}$, $\varphi_2(x) = -w_1^{-1}w_2$ are determined from the system of ordinary differential equations

$$\varphi_1'' = \frac{6}{\lambda}\varphi_1^3, \quad \varphi_2'' = \frac{6}{\lambda}\varphi_1^2\varphi_2. \tag{51}$$

If, for example, in equation (48) $p = q = 0$, then the solution of equation (48) is the function $d = u^{1/3}$, and from (49) it follows $F(u) = u^{-2/3}$. Then a couple of the functions $\varphi_1(x) = x^{-1}$, $\varphi_2(x) = x^2$ are the particular solution of (51), and so the function

$$u^{1/3} = x^{-1}t + cx^2 \ ,$$

is a solution of the equation

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial x}\left(u^{-2/3}\frac{\partial u}{\partial x}\right),$$

(c is an arbitrary constant).

This solution has been obtained in the paper [15] by means of the conditional symmetry technique, while the solution $d(u) = \varphi_1(x)t + \varphi_2(x)$ of equation (50), which has been considered above, can not be obtained by this approach. Note that the result of investigation of equation (50) with $a = 1$ (see paper [15]) by the method of conditional symmetry shows that this method is cumbersome and ineffective to construct exact solutions of equation (1). It should also be mentioned that solutions (35), (36), (43), and (44), constructed in this paper, can not be obtained by the classical symmetry analysis approach.

4 Conclusions

To find solutions of nonlinear partial differential equations often it need to use simultaneously ansatzes of types (3) and (4), as well ansatzes which are obtained from (3) as a result of permutation of the variables u, t and x. So, equation (17) has been selected as the equation that admits ansatz (5), where $d(u)$ ia the solution of equation (13). Local transformation (18) reduces equation (17) to equation (19) with a square nonlinearity, solutions of which are searched with using ansatz (3).

For equation (1) corresponding to the parameter $a = 1$ it is efficiently to use the ansatz

$$x = w_1(u)d(t) + w_2(u), \tag{52}$$

which reduces equation (1) to the system

$$2w_1(w_1')^2 - (w_1)^2 w_1'' = -w_1''F + w_1'F',$$

$$2w_1w_1'w_2' - (w_1)^2 w_2'' = -w_2''F + w_2'F'.$$

The solution of this system is the pair of the functions $w_1 = F(u)^{1/2}$, $w_2 = w_2(u)$, at that w_2 is an arbitrary smooth function. The function $d = d(t)$ in (52) should be linear, and therefore we can assume that $d(t) = t$. Thus, the function

$$x = F(u)^{1/2}t + w_2(u),$$

with an arbitrary smooth function w_2 is a solution of equation (1) for $a = 1$. This solution has been obtained in [15] by the method of conditional symmetries.

To construct solutions of nonlinear partial differential equations it can be used the more general ansatz

$$p(u) = \sum_{i=1}^{m} w_i(t)a_i(x) + f(x,t), \qquad m \geq 1, \tag{53}$$

as well as ansatzes, coming from (53) by permutation of the variables u, t and x. The function in (53) can be set a priori, and the other functions $w_i(t)$, $a_i(x)$, $f(x,t)$ can be defined as in the case $p(u) = u$. Let us note that all solutions of [4–6] can be reobtained with usage the ansatz (53).

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