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A CONVERGENCE CLASS FOR ENTIRE DIRICHLET SERIES OF SLOW GROWTH

Dedicated to the 70th anniversary of Prof. A. A. Gol'dberg

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For an entire Dirichlet series with nonnegative increasing to $+\infty$ exponents a connection between the growth of maximum modulus and the behaviour of coefficients is established in the terms of certain convergence class.

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Для целого ряда Дирихле с неотрицательными возрастающими к $+\infty$ показателями в терминах определенного класса сходимости установлена связь между ростом максимума модуля и поведением коэффициентов.

Let $0 = \lambda_0 < \lambda_n \uparrow +\infty$ and

$$F(s) = \sum_{n=0}^{\infty} a_n \exp(s\lambda_n), \quad s = \sigma + it, \tag{1}$$

be an entire (absolutely convergent in $\mathbb{C}$) Dirichlet series. We say that this series of is of slow growth if $\ln M(\sigma) = (1+o(1))\sigma l(\sigma)$ $(\sigma \to +\infty)$, where $M(\sigma) = \sup\{|F(\sigma+it)| : t \in \mathbb{R}\}$ and l is a slowly increasing function, i.e. l is a positive increasing to $+\infty$ function on $[x_0, +\infty)$ such that $xl'(x)/l(x) \to 0$, $x \to \infty$. Further, for simplicity, we consider that $x_0 = 1$. Generalizing a result of Valiron [1] on the belonging of an entire function of finite order to classical convergence class, Kamthan [2] indicated conditions on the exponents and the coefficients of series (1) in order that $\int_0^\infty \exp\{-\varrho\sigma\} \ln M(\sigma) d\sigma < +\infty$. This results is generalized in [3–4], where generalized convergence classes are introduced and studied. Here we supplement the results from [3–4] for entire Dirichlet series of slow growth.

Let α be a slowly increasing function. We say that Dirichlet series (1) belongs to a convergence α-class provided

$$\int_1^\infty \frac{\ln M(\sigma) d\sigma}{\sigma^2 \alpha(\sigma)} < +\infty, \quad \text{if} \quad \int_1^\infty \frac{dt}{t\alpha(t)} < +\infty, \tag{2}$$

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and

$$\int_1^\infty \frac{d\sigma}{\alpha(\sigma)\ln M(\sigma)} < +\infty, \quad \text{if} \quad \int_1^\infty \frac{dt}{t\alpha(t)} = +\infty. \tag{3}$$

Suppose that $\ln n = O(\lambda_n)$, $n \to \infty$. Then [5, p. 184] there exists $K > 0$ and $\tau > 0$ such that $M(\sigma) \le K\mu(\sigma+\tau)$ for all $\sigma \ge 1$, where $\mu(\sigma) = \max\{|a_n|\exp\{\sigma\lambda_n\} : n \ge 0\}$ is the maximal term of series (1). Hence in view of the Cauchy inequality $\mu(\sigma) \le M(\sigma)$ [5, c. 125] it follows that in (2) and (3) we can put $\ln\mu(\sigma)$ instead of $\ln M(\sigma)$.

Let $\nu(\sigma) = \max\{n : |a_n|\exp\{\sigma\lambda_n\} = \mu(\sigma)\}$ be the central index of series (1). Then [5, c. 182] $\ln\mu(\sigma) - \ln\mu(\sigma_0) = \int_{\sigma_0}^{\sigma} \lambda_{\nu(t)}dt$, whence in view of nondecreasing of $\lambda_{\nu(\sigma)}$ we easily obtain the following estimates $(\sigma/2)\lambda_{\nu(\sigma/2)} \le \ln\mu(\sigma) - \ln\mu(\sigma_0) \le \sigma\lambda_{\nu(\sigma)}$. Thus, *Dirichlet series (1) belongs to the convergence α-class iff*

$$\int_1^\infty \frac{\lambda_{\nu(\sigma)}d\sigma}{\sigma\alpha(\sigma)} < +\infty, \quad 1 \le \sigma_0 < +\infty, \quad \text{if} \quad \int_1^\infty \frac{dt}{t\alpha(t)} < +\infty, \tag{4}$$

and

$$\int_1^\infty \frac{d\sigma}{\sigma\alpha(\sigma)\lambda_{\nu(\sigma)}} < +\infty, \quad \text{if} \quad \int_1^\infty \frac{dt}{t\alpha(t)} = +\infty. \tag{5}$$

Finally, let a_n^o be the coefficients of Newton majorant of Dirichlet series (1) and $\varkappa_n^o = \frac{\ln a_n^o - \ln a_{n+1}^o}{\lambda_{n+1}-\lambda_n}$. Then [5, p. 180–183] $|a_n| \le a_n^o$, $\varkappa_n^0 \nearrow +\infty$, and if $\varkappa_{n-1}^o \le \sigma < \varkappa_n^o$ then $\lambda_{\nu(\sigma)} = \lambda_n$. For simplicity, we suppose that $\varkappa_0^o \ge 1$. Therefore,

$$\int_1^\infty \frac{\lambda_{\nu(\sigma)}d\sigma}{\sigma\alpha(\sigma)} + \text{const} = \sum_{n=1}^\infty \int_{\varkappa_{n-1}^o}^{\varkappa_n^o} \frac{\lambda_{\nu(\sigma)}d\sigma}{\sigma\alpha(\sigma)} =$$
$$= \sum_{n=1}^\infty \lambda_n \int_{\varkappa_{n-1}^o}^{\varkappa_n^o} \frac{d\sigma}{\sigma\alpha(\sigma)} = \sum_{n=1}^\infty \lambda_n(\beta_1(\varkappa_{n-1}^o) - \beta_1(\varkappa_n^o))) =$$
$$= \sum_{n=1}^\infty (\lambda_n - \lambda_{n-1})\beta_1(\varkappa_{n-1}^o) + \text{const}, \quad \beta_1(x) = \int_x^\infty \frac{d\sigma}{\sigma\alpha(\sigma)}. \tag{6}$$

By analogy,

$$\int_1^\infty \frac{d\sigma}{\sigma\alpha(\sigma)\lambda_{\nu(\sigma)}} + \text{const} = \sum_{n=1}^\infty \frac{1}{\lambda_n}\int_{\varkappa_{n-1}^o}^{\varkappa_n^o} \frac{d\sigma}{\sigma\alpha(\sigma)} = \sum_{n=1}^\infty \frac{1}{\lambda_n}(\beta_2(\varkappa_n^o) - \beta_2(\varkappa_{n-1}^o)) =$$
$$= \sum_{n=1}^\infty \left(\frac{1}{\lambda_{n-1}} - \frac{1}{\lambda_n}\right)\beta_2(\varkappa_{n-1}^o) + \text{const}, \quad \beta_2(x) = \int_1^x \frac{d\sigma}{\sigma\alpha(\sigma)}. \tag{7}$$

Thus, we need to investigate the convergence of the last series in (6) and (7).

From the definition of $\varkappa_{n-1}^o$ it follows that

$$\frac{1}{\lambda_n}\ln\frac{1}{a_n^0} = \frac{\varkappa_0^o\lambda_1^* + \cdots + \varkappa_{n-1}^o\lambda_n^*}{\lambda_1^* + \cdots + \lambda_n^*}, \quad \lambda_n^* = \lambda_n - \lambda_{n-1}, \tag{8}$$

whence we obtain the inequality $\frac{1}{\lambda_n}\ln\frac{1}{a_n^0} \le \varkappa_{n-1}^o$. Therefore, if

$$\sum_{n=1}^\infty (\lambda_n - \lambda_{n-1})\beta_1\left(\frac{1}{\lambda_n}\ln\frac{1}{a_n^0}\right) < +\infty, \tag{9}$$

then, in view of decrease of the function β_1, relation (4) holds, and if (5) holds then in view of increase of the function β_2 we have

$$\sum_{n=1}^{\infty}\left(\frac{1}{\lambda_{n-1}}-\frac{1}{\lambda_n}\right)\beta_2\left(\frac{1}{\lambda_n}\ln\frac{1}{a_n^0}\right)<+\infty. \tag{10}$$

In order to obtain a converse to the first assertion, we use the following

Lemma [4]. *Let $p>1$, $q=\frac{p}{p-1}$ and f be a positive function on (A,B), $-\infty\le A<B\le+\infty$, such that the function $f^{1/p}$ is convex on (A,B). Let (λ_n^*) be a sequence of positive numbers, (a_n) be a sequence of numbers from (A,B) and $A_n=\frac{\lambda_1^*a_1+\cdots+\lambda_n^*a_n}{\lambda_1^*+\cdots+\lambda_n^*}$. Finally, let (μ_n) be a positive nonincreasing sequence. Then*

$$\sum_{n=1}^{\infty}\mu_n\lambda_n^*f(A_n)\le q^p\sum_{n=1}^{\infty}\mu_n\lambda_n^*f(a_n). \tag{11}$$

Since the function α is increasing to $+\infty$, then it is easy to show that the function $\beta_1^{1/2}(x)$ is convex on an interval where $\alpha(x)\ge1$. Therefore, if we put in Lemma $p=2$, $\mu_n\equiv1$ and $\lambda_n^*=\lambda_n-\lambda_{n-1}$, then in view of (8) we have

$$\sum_{n=1}^{\infty}(\lambda_n-\lambda_{n-1})\beta_1\left(\frac{1}{\lambda_n}\ln\frac{1}{a_n^0}\right)\le4\sum_{n=1}^{\infty}(\lambda_n-\lambda_{n-1})\beta_1(\varkappa_{n-1}^o).$$

Thus, in view of (6) we have proved that *(4) holds iff (9) holds.*

In order to obtain an converse to the second assertion, we remark that in view of (8) $\frac{1}{\lambda_n}\ln\frac{1}{a_n^0}\ge\frac{\lambda_n-\lambda_{n-1}}{\lambda_n}\varkappa_{n-1}^o$ and

$$\beta_2(e^{2x})=\int_0^{2x}\frac{dt}{\alpha(e^t)}=\int_0^{x}\frac{dt}{\alpha(e^t)}+\int_x^{2x}\frac{dt}{\alpha(e^t)}=$$
$$=\int_0^{x}\frac{dt}{\alpha(e^t)}+\int_0^{x}\frac{dt}{\alpha(e^{t+x})}\le2\int_0^{x}\frac{dt}{\alpha(e^t)}=2\beta_2(e^x).$$

Therefore,

$$\beta_2(\varkappa_{n-1}^o)\le\beta_2\left(\exp\left\{\ln\left(\frac{1}{\lambda_n}\ln\frac{1}{a_n^0}\right)+\ln\frac{\lambda_n}{\lambda_n-\lambda_{n-1}}\right\}\right)\le$$
$$\le\beta_2\left(\exp\left\{2\max\left\{\ln\left(\frac{1}{\lambda_n}\ln\frac{1}{a_n^0}\right),\ln\frac{\lambda_n}{\lambda_n-\lambda_{n-1}}\right\}\right\}\right)\le$$
$$\le2\beta_2\left(\max\left\{\ln\left(\frac{1}{\lambda_n}\ln\frac{1}{a_n^0}\right),\ln\frac{\lambda_n}{\lambda_n-\lambda_{n-1}}\right\}\right)=$$
$$=2\max\left\{\beta_2\left(\frac{1}{\lambda_n}\ln\frac{1}{a_n^0}\right),\beta_2\left(\ln\frac{\lambda_n}{\lambda_n-\lambda_{n-1}}\right)\right\}\le$$
$$\le2\left(\beta_2\left(\frac{1}{\lambda_n}\ln\frac{1}{a_n^0}\right)+\beta_2\left(\ln\frac{\lambda_n}{\lambda_n-\lambda_{n-1}}\right)\right).$$

Hence if

$$\sum_{n=1}^{\infty}\left(\frac{1}{\lambda_{n-1}}-\frac{1}{\lambda_n}\right)\beta_2\left(\ln\frac{\lambda_n}{\lambda_n-\lambda_{n-1}}\right)<+\infty, \tag{12}$$

then from (10) we obtain the convergence of the last series in (7).

Thus, we have proved that *if the sequence* (λ_n) *satisfies condition (12), then (5) holds iff (10) holds.*

Remarking that $|a_n| \le a_n^o$ for all $n \ge 0$ and $|a_n| = a_n^o$ for all $n \ge 0$ provided $\varkappa_n = \frac{\ln|a_n| - \ln|a_{n+1}|}{\lambda_{n+1} - \lambda_n} \nearrow +\infty$, we therefor come to the following

Theorem. *Let the exponents of entire Dirichlet series (1) satisfy the condition* $\ln n = O(\lambda_n)$ $(n \to \infty)$, *and* α *be a slowly increasing function. Then:*

i) if $\int_1^\infty \frac{dt}{t\alpha(t)} < +\infty$ *then in order that Dirichlet series belong to the convergence* α*-class, it is necessary and in the case when* $\varkappa_n \nearrow +\infty$ *it is sufficient that condition (9) hold with* $|a_n|$ *instead* a_n^0*;*

ii) if $\int_1^\infty \frac{dt}{t\alpha(t)} = +\infty$ *then in order that Dirichlet series belong to the convergence* α*-class, in the case* $\varkappa_n \nearrow +\infty$ *it is necessary and in case when the exponents satisfy condition (12) it is sufficient that condition (10) hold with* $|a_n|$ *instead* a_n^0*.*

Remark 1. In the proof of necessity of condition (9) in the first assertion of Theorem the condition $\ln n = O(\lambda_n)$ $(n \to \infty)$ is not used. In the proof of sufficiency we can replace this condition by the following condition

$$\varlimsup_{n\to\infty} \frac{\ln n}{-\ln|a_n|} = h < 1. \tag{13}$$

Indeed, if (13) holds then for all $\sigma \ge 0$ the following estimate is true [6, c. 23] $M(\sigma) \le A_0(\varepsilon)\mu\left(\frac{\sigma}{1-h-\varepsilon}\right)$ for every $\varepsilon \in (0, 1-h)$ and some $A_0(\varepsilon) > 0$, and since α is slowly increasing then, in view of the Cauchy inequality, the integrals $\int_1^\infty \frac{\ln M(\sigma)}{\sigma^2\alpha(\sigma)}d\sigma$ and $\int_1^\infty \frac{\ln\mu(\sigma)}{\sigma^2\alpha(\sigma)}d\sigma$ are either convergent or divergent simultaneously. The further proof of sufficiency is analogous to that given above.

Further, if $\int_1^\infty \frac{\ln\mu(\sigma)}{\sigma^2\alpha(\sigma)}d\sigma < +\infty$ then using L'Hospital rule and the slow increase of α we obtain for all enough large σ

$$\frac{1}{2} \ge \int_\sigma^\infty \frac{\ln\mu(x)dx}{x^2\alpha(x)} \ge \ln\mu(\sigma)\int_\sigma^\infty \frac{d\sigma}{x^2\alpha(x)}dx = (1+o(1))\frac{\ln\mu(\sigma)}{\sigma\alpha(\sigma)}, \quad \sigma \to +\infty,$$

that is $\ln\mu(\sigma) \le \sigma\alpha(\sigma)$, $\sigma \ge \sigma_0$, and $\ln|a_n| \le \sigma\alpha(\sigma) - \sigma\lambda_n$ for all $n \ge 0$ and $\sigma \ge \sigma_0$. Putting here $\sigma = \varphi(\lambda_n)$, where $\varphi(x)$ is a solution of the equation $\alpha(\sigma) + \sigma\alpha'(\sigma) = x$, we have for $n \ge n_0$ the inequality $\ln|a_n| \le -\varphi(\lambda_n)\alpha'(\varphi(\lambda_n))$, that is (13) holds provided

$$\varlimsup_{n\to\infty} \frac{\ln n}{\varphi(\lambda_n)\alpha'(\varphi(\lambda_n))} < 1. \tag{14}$$

Thus, the condition $\ln n = O(\lambda_n)$ $(n \to \infty)$ in the first assertion of Theorem can be replaced by condition (14).

Remark 2. The condition $\ln n = O(\lambda_n)$ $(n \to \infty)$, is used only to prove the necessity of condition (10). We can replace it by condition (13), but it is impossible to find a condition similar to (15) because the convergence of integral $\int_1^\infty \frac{d\sigma}{\alpha(\sigma)\ln\mu(\sigma)}$ for $\ln\mu(\sigma)$ it can yield only an estimate from below.

Condition (12) appeared as a result of the applied method. We could not find out whether this condition is superfluous.

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