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ESTIMATION OF CRACK PARAMETERS BY THE LEVEL OF THE VIBRATION RESPONSE NON-LINEARITY OF A CRACKED BEAM AT SUPER- AND SUBHARMONIC RESONANCES

The proposed procedure of crack location and size estimation is based on the determination of the vibration response non-linearity around the superharmonic resonance of order 2/1 and subharmonic resonance of order 1/2 when the driving force is applied at different points.

Keywords: damage diagnosis, closing crack, damping, super- and subharmonic resonance vibrations, driving force parameters.

Introduction. Fatigue cracks are the most widespread damage of dynamically loaded structural elements. Such cracks periodically close and open in the process of cyclic deformation of a body, leading to the instantaneous change of its stiffness. Usually the change of stiffness is modeled by the piece-wise linear characteristic of the restoring force [1] or by the specific modification of the driving force [2].

A closing crack causes the dynamic behavior of vibrating system to be significantly non-linear, creating a series of fundamental difficulties with regard to determining analytical solutions [1, 2]. Numerical investigations of forced vibrations of beams with a closing crack [3, 4] have demonstrated that the main distinctive features of such a vibration system are the manifestation of effects associated with non-linearity, namely the presence of super- and subharmonic resonances and significant non-linearity of the vibration response at super- and subharmonic resonances (so called non-linear resonances) of different orders. The reason of the considerable non-linear distortions of the vibration response at non-linear resonances is the fact that at these regimes the spectrum of vibration response contains the harmonic the frequency of which coincides with the frequency of the principal resonance. The amplitude of this harmonic exceeds considerably the amplitudes of other harmonics. That is why it was called the dominant one and was used as an indicator of crack presence [5].

The sensitivity of non-linear effects to the crack presence exceeds many times (one or even two orders of magnitude) the sensitivity of natural frequencies and mode shapes [3–5]. This opens up the perspective of using super- and subharmonic resonance regimes for the detection of fatigue cracks.

Manifestation of the non-linear effects depends not only on the crack parameters (that is crack size and location) but also on the level of damping in a vibrating system [5, 6] and on the power of excitation [6, 7]. The data of direct experimental investigations [8–10] attest that the fatigue crack growth is accompanied by a considerable increase of damping characteristic of specimens.
The aim of the work was to develop the strategy of damage detection which consists in determination of the non-linearity of vibration response of a structure at super- and subharmonic resonances in condition of load application in different points of a structure.

**Model of a cracked beam.** The presented mathematical model of a cantilever beam with an edge transverse closing crack is based on the Finite Element (FE) model proposed in the work [3]. While as the crack is closed and its interfaces are completely in contact with each other, the dynamic response can be determined directly as that one of the intact beam. However, while as the crack is opened the stiffness matrix of the cracked element should be introduced in replacement at the appropriate rows and columns of the general stiffness matrix. It was suggested that the system has the piecewise-linear characteristic of the restoring force. The vibration of a cracked beam was described by two linear differential equations in normalized co-ordinates:

\[
\begin{align*}
[I][\ddot{q}] + [\Lambda][\dot{q}] + [\omega^2][q] &= [R]F(t), \\
[I][\ddot{q}] + [\Lambda_d][\dot{q}] + [\omega^2_d][q] &= [R_d]F(t),
\end{align*}
\]

where the matrices of a beam with a closed crack are shown without subscripts and the matrices of a beam with an open crack are shown with subscript d (damaged beam); \([I]\) is the mass matrix (unitary); \([\omega^2]\) is the stiffness matrix; \([\Lambda]\) is the damping matrix; \([R]\) is the external load vector; \(F(t)\) is the external load function; \({q}\) is the normal coordinates. The first equation describes the vibration of a beam with a closed crack and the second one - of a beam with an open crack. These equations were solved with the Runge-Kutta method proceeding step-by-step in time.

In the process of the solution of Eqs. (1) the check of the crack state is realized by the comparison of the angles of rotation in the nodes from the right and from the left of the cracked element.

Since the normal coordinates express the amount of each mode shape in the vibration process and these mode shapes are different for the damaged and undamaged beam, then in the instants of crack closure or opening at the same \({q}\) the displacements of the beam are different, that is the beam is in two positions simultaneously. To avoid this contradiction it is necessary to bring to conformity the normal coordinates with the real displacements of the beam for each time step when the change of crack state takes place. If during the i-th time step the state of crack changed, then the initial conditions for the next \((i+1)\) time step should be calculated by the following formulas:

\[
\begin{align*}
\{q_{i+1}\} &= [\Phi_d]^{-1}[\Phi][q_i], \\
\{q_{i+1}\} &= [\Phi]^{-1}[\Phi_d][q_i],
\end{align*}
\]

where \([\Phi]\) is the matrix of mode shapes of the beam with a closed crack; \([\Phi_d]\) is the matrix of mode shapes of the beam with an open crack. Formula (2) is used for the moment of crack opening and formula (3) - of crack closure. The mesh with a local concentration in the area of crack location is used for the increase of accuracy of...
the determination of crack location (Fig. 1). To calculate the equivalent stiffness of the cracked element the strain energy balance method is used [5]

\[
P_0 \sin \omega t
\]

Fig. 1. The finite element mesh for the beam.

Preliminary experimental appraisal of the model showed its ability to predict accurately the change of axial and bending natural frequencies of beams with an edge fatigue crack: the relative error did not exceed 4%.

Further validation of the model confirmed that it can reveal the non-linear resonances, in particular, the superharmonic resonance of order 2/1 and subharmonic resonance of order 1/2. Calculations were made for the beam with the following parameters: \( L = 0.23 \) m; \( L_c/L = 0.1; h = 0.02 \) m; \( b = 0.004 \) m; \( E = 206 \) GPa; \( \rho = 7850 \) kg/m\(^3\); \( \delta = 0.01 \) (\( \delta \) is the logarithmic decrement). In this case the amplitudes of super- and subharmonic resonances are respectively 144 and 299 times less than the amplitudes of principal resonance. The distinction between the amplitudes of principal and non-linear resonances decreases with the growth of crack size and damping level. Anyway, the amplitudes of non-linear resonances are very small to use there change for the detection of crack. Besides the results of calculations showed that the width of resonance curve at the principal and subharmonic of order 1/2 resonances are comparable by the value, but superharmonic resonance of order 2/1 is about three times more narrow then principal one.

**Experimental procedure and results of experiment.** The results of two specimens test were used to validate the presented above FE model of a beam. The dimensions of the specimens and mechanical properties of the materials are shown in Table 1, where \( L \) is the length of the specimens; \( h \) and \( b \) are the height and width of the cross-section, respectively; \( E \) is Young’s modulus; \( \rho \) is the density; \( \sigma_{l} \) is the fatigue limit. The parameters of crack size (\( a \)) and location (\( L_c \)), as well as the location of accelerometer (\( L_{ac} \)) and driving force application points (\( L_P \)) are given in Table 2.

**Table 1**

<table>
<thead>
<tr>
<th>Material of the specimens</th>
<th>( L ) [mm]</th>
<th>( h ) [mm]</th>
<th>( b ) [mm]</th>
<th>( E ) [GPa]</th>
<th>( \rho ) [kg/m(^3)]</th>
<th>( \sigma_{l} ) [MPa]</th>
</tr>
</thead>
<tbody>
<tr>
<td>VT-3</td>
<td>230</td>
<td>20</td>
<td>4</td>
<td>110</td>
<td>4480</td>
<td>480</td>
</tr>
<tr>
<td>Steel 3</td>
<td>230</td>
<td>19.5</td>
<td>4</td>
<td>200</td>
<td>7800</td>
<td>190</td>
</tr>
</tbody>
</table>

**Table 2**

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>( L_c ) [mm]</td>
<td></td>
</tr>
<tr>
<td>( a ) [mm]</td>
<td></td>
</tr>
<tr>
<td>( L_{ac} ) [mm]</td>
<td></td>
</tr>
<tr>
<td>( L_P ) [mm]</td>
<td></td>
</tr>
</tbody>
</table>

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Table 2

Parameters of tests at non-linear resonances

<table>
<thead>
<tr>
<th>Specimens</th>
<th>Non-linear resonance</th>
<th>$a$</th>
<th>$L_a$</th>
<th>$L_{ac}$</th>
<th>$L_{p1}$</th>
<th>$L_{p2}$</th>
<th>$\delta$</th>
</tr>
</thead>
<tbody>
<tr>
<td>VT-3</td>
<td>superharmonic</td>
<td>2.1</td>
<td>11</td>
<td>223</td>
<td>180</td>
<td>96</td>
<td>0.009</td>
</tr>
<tr>
<td></td>
<td>subharmonic</td>
<td>4.0</td>
<td>11</td>
<td>223</td>
<td>181</td>
<td>99</td>
<td>0.011</td>
</tr>
<tr>
<td>Steel 3</td>
<td>superharmonic</td>
<td>5.1</td>
<td>14</td>
<td>223</td>
<td>184</td>
<td>99</td>
<td>0.020</td>
</tr>
<tr>
<td></td>
<td>subharmonic</td>
<td>8.0</td>
<td>14</td>
<td>223</td>
<td>182</td>
<td>99</td>
<td>0.020</td>
</tr>
</tbody>
</table>

The tests were performed using the experimental set-up KD-1M [5]. This set-up has been devised for the determination of damping characteristic and high-cycle fatigue testing of specimens, and for the modal analysis of specimens. The specimens were fixed in the rigid frame (Fig. 2). The electromagnetic system was used for the excitation of the first bending mode of vibration. This system consists of two electromagnets, computer-based waveform generator and power amplifier. The special ferromagnetic plates were attached to the specimens to create the interaction between the specimen and electromagnets. Fatigue cracks were grown from sharp concentrators in fatigue tests. The measurements of the cracks depth were executed by an optical microscope, the absolute error being ±0.1 mm. The damping characteristic of cracked specimens $\delta$ (logarithmic decrement of vibration (LDV)) was determined by the free damped oscillation method.

Fig. 2. Schematic sketch of the specimen and of the loading system.

The tests of titanium alloy VT-3 and carbon Steel 3 specimens showed that the origination of the crack even small causes the increase of LDV in several times. These data were used in the FE model of the cantilever beam in calculations of non-linear effects (Fig. 3).
Even in the case of small unharmonicity of the driving force the so-called pseudo-superharmonic resonance may appear [5]. To avoid this regime of vibration the special algorithm of signal distortion by the waveform generator of the excitation system was used. The initial signal was distorted in such a way that its subsequent distortions by the power amplifier and electromagnets compensate the initial one and, as a consequence, the spectrum of driving force practically does not contain the high harmonic the frequency of which coincides with the frequency of the principal resonance or, at least, its amplitude is sufficiently small.

On excitation of the subharmonic resonance there is no need of such a thorough monitoring of the driving force non-linearity.
The experiments at super- and subharmonic resonances were carried out in the range of stress amplitudes $c_a = 5...15$ MPa. As can be seen from Fig. 3, the non-linearity around the non-linear resonances is dependent on the point of force application. The influence of the force application point is qualitatively different at super- and subharmonic resonances: in the first case the non-linearity of acceleration response is lower at upper point of force application than at middle one, in the second case – high.

Quantitative discrepancy between the results of calculations and the data of experiments for the Steel 3 specimen in the case of superharmonic resonance may be attributed to the acuteness of the resonance peak: it was impossible to change the frequency of the excitation force with the required discreteness. In addition, the FE model makes it possible to apply the force only in nodes, therefore the conditions of specimens loading may be reproduced with certain error.

The tests revealed the main distinctive feature of subharmonic resonance. The power necessary for the excitation of this regime of vibration exceeds considerably that one for the excitation of superharmonic resonance. Under the insufficient power of excitation the subharmonic resonance does not appear even in the presence of big crack. These results support the conclusion of analytical investigations [6, 7]. For the alloy VT-3 specimen in case of $L_p = 99$ mm the subharmonic resonance was not excited at all in spite of the fact that the utmost power of the equipment was used. The FE model does not reveal this feature of subharmonic resonance and therefore the model requires further improvement.

Most likely, this is the reason of essential quantitative discrepancy between the results of calculations and experiments for the VT-3 alloy specimen. The subharmonic resonance has some other features. In case of small crack it appears and disappears abruptly. Besides subharmonic regime of vibration at certain conditions is extremely unsteady.

The experiments showed that the shakers and other systems of excitation which use the permanent magnetic biasing coil are inefficient for the excitation of non-linear resonances because the dominant harmonic in the vibration response spectrum is essentially suppressed by such shakers.

As can be judged from the above results, the change of non-linearity of the vibration response around the super- and subharmonic resonances is much more sensitive to the crack presence than the change of the amplitude of vibration at non-linear resonances.

**Effect of driving force application point: results of calculations.** As the FE model describes accurately enough the non-linear effects, it was used for the investigation of the effect of the driving force application point on the non-linearity of acceleration response (in this calculation it was related to the end of the beam). The parameters of the beam were the same as in the section Model of a cracked beam but two crack locations were considered ($L_s/L = 0.1$ or $0.5$).

The procedure of calculations was as follows: the force was applied to one node of FE model after another (Fig. 1) and at that the spectral analysis of the acceleration response of the beam end was executed. As can be seen from Fig. 4, the calculated non-linearity of acceleration response of the cracked beams at both non-linear resonances significantly depends on the driving force application point. The sharp local break of these curves distinctly reveals the crack location. The level of non-linearity makes it possible to judge about the crack size on condition...
that the damping characteristic of the cracked beam is known.

![Graphs showing amplitudes of dominant harmonics vs. the coordinate of the force application point: superharmonic of order 2/1 (a) and subharmonic of order 1/2 (b) resonances.](image)

**Fig. 4.** The amplitudes of dominant harmonics vs. the co-ordinate of the force application point: superharmonic of order 2/1 (a) and subharmonic of order 1/2 (b) resonances.

The different location of sensor along the beam (Fig. 1) practically does not effect on the vibration response non-linearity. So in the case of the proposed method of damage detection the best sensor location is conditioned mainly by the consideration of obtaining the most level of signal and thereby to reduce the noise-to-signal ratio.

At the frequency of subharmonic resonance the non-linearity of vibration response shows up only if the crack reaches a certain definite size. Since that value the damage characteristic shows high sensitivity to the crack presence. At superharmonic resonance the increase of dominant harmonic with crack grows is less intensive but it takes place since minimal crack size.

**Conclusions.** The non-linearity of vibration response around the sub- and superharmonic resonances of beams is very sensitive to the presence of closing crack. This non-linearity, as was shown both experimentally and numerically, is strongly dependent on the crack size and location, damping level, driving force application point and driving force asymmetry.

The excitation of vibration of a beam by the application of concentrated force to different its points resulted in the essential change of the level of vibration response non-linearity at super- and subharmonic resonances. The abrupt change of this non-linearity indicates the crack location. At the same time the level of vibration response non-linearity is directly proportional to the crack size. Consequently, the measurements of the vibration response non-linearity around the super- and/or subharmonic resonances applying the driving force in several points along the beam enable to determine the crack parameters by the proposed FE
The accuracy of the crack size determination depends on the accuracy of the damping characteristic estimation.

The proposed experimental procedure made it possible to solve a problem of excitation of super- and subharmonic resonances. The experimental investigations revealed that for the excitation of subharmonic resonance it is necessary to apply much bigger power than for the excitation of superharmonic one. At the same time the advantage of subharmonic resonance is that it does not need such a thorough monitoring of the driving force non-linearity as in case of superharmonic one.

One more damage detection procedure is proposed. Its essence consists in the artificial variation of the crack state by the addition of the static component to the harmonic driving force. The change of vibration response non-linearity at super- or subharmonic resonances in the course of such a variation can serve as an indicator of crack presence. The advantage of the procedure is that no preliminary information on the vibration response of a structure in an intact state is required.

Резюме

Предложен метод оценки размеров и местоположения трещины, основанный на определении уровня нелинейности колебаний в окрестности супергармонического порядка 2/1 и субгармонического порядка 1/2 резонансов.

Ключевые слова: диагностика повреждения, закрывающаяся трещина, демпфирование, супер- и субгармонические резонансные колебания, параметры вынуждающей силы.

Резюме

Запропоновано метод оцінки розмірів і місцезнаходження тріщини, який грунтується на визначені рівня нелінійності коливань в окопі супергармонічного порядку 2/1 і субгармонічного порядку 1/2 резонансів.

Ключові слова: діагностика пошкодження, тріщина, що закривається, демпфування, супер- і субгармонічні резонансні коливання, параметри змушувальної сили.

5. Bovsunovsky A. P., Surace C. Considerations regarding superharmonic


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