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EXACT SOLUTIONS OF THE NONLINEAR EQUATION $u_{tt} = a(t)uu_{xx} + b(t)u_x^2 + c(t)u$

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We determine ansätzes that reduce the equation $u_{tt} = a(t)uu_{xx} + b(t)u_x^2 + c(t)u$ to a system of two ordinary differential equations. It is also shown that the problem of construction of exact solutions of this equation of the form $u = \mu_1(t)x^2 + \mu_2(t)x^\alpha$, $\alpha \in \mathbf{R}$, reduces to the integration of a system of linear equations $\mu_1'' = \Phi_1(t)\mu_1$, $\mu_2'' = \Phi_2(t)\mu_2$, where $\Phi_1(t)$ and $\Phi_2(t)$ are arbitrary given functions.

1. Introduction

In the present paper, we consider the problem of construction of exact solutions with generalized separation of variables for a nonlinear equation

$$\frac{\partial^2 u}{\partial t^2} = a(t)u\frac{\partial^2 u}{\partial x^2} + b(t)\left(\frac{\partial u}{\partial x}\right)^2 + c(t)u, \tag{1}$$

where $a = a(t)$, $b = b(t)$, and $c = c(t)$ are functions of t. Equations of this type are often encountered in the problems of wave and gas dynamics.

In the general case, Eq. (1) has the exact solution [5]

$$u = \mu_2(t)x^2 + \mu_1(t)x + \mu_0(t).$$

Special cases of this equation were studied in [1, 2, 4, 5]. A method for the construction of the exact solutions with generalized separation of variables

$$u = \sum_{i=1}^{n} \psi_i(t)\varphi_i(x)$$

was proposed in [1–4]. This method is based on finding finite-dimensional subspaces invariant under the action of the nonlinear differential operator corresponding to Eq. (1). In this case, the system of coordinate functions $\varphi_i(x)$ is given *a priori* and the functions $\psi_i(t)$ are determined by the method of undetermined coefficients.

In [6, 7], for the construction of the exact solutions of Eq. (1), we use the following ansatz:

$$u = d(x)\omega(t) + f(t,x), \tag{2}$$

where the unknown functions $d = d(x)$, $\omega = \omega(t)$, and $f = f(t,x)$ are determined from the condition that ansatz (2) reduces Eq. (1) to an ordinary differential equation with the unknown function $\omega = \omega(t)$.

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In the present paper, with the help of ansatz (2), we determine the systems of coordinate functions for the exact solutions of Eq. (1). This reduces the problem of construction of solutions to the integration of a system of ordinary differential equations. Thus, in the case

$$a(t) = \frac{\alpha}{1-\alpha} b(t), \qquad \alpha \in \mathbf{R}, \quad \alpha \neq 1, 2, 3,$$

the functions x^2 and x^α form a system of coordinate functions. At the same time, for $a(t) = -2b(t)$, the functions x^2 and $x^2 \ln|x|$ form a system of coordinate functions. In particular, it is shown that the problem of finding the exact solutions of Eq. (1) of the form

$$u = \mu_1(t)x^2 + \mu_2(t)x^\alpha$$

is reduced to the integration of the following system of ordinary differential equations:

$$\mu_1'' = \Phi_1(t)\mu_1, \qquad \mu_2'' = \Phi_2(t)\mu_2, \tag{3}$$

where $\Phi_1(t)$ and $\Phi_2(t)$ are arbitrary given functions. In numerous cases, system (3) can be integrated.

2. Reduction of Eq. (1) to a System of Ordinary Differential Equations

We start from a special case of Eq. (1):

$$\frac{\partial^2 u}{\partial t^2} = a(t)u\frac{\partial^2 u}{\partial x^2} + b(t)\left(\frac{\partial u}{\partial x}\right)^2. \tag{4}$$

For the construction of the exact solutions of Eq. (4), we use ansatz (2) that reduces Eq. (4) to the following equation:

$$\omega'' d - \omega(af_{xx}d + 2bf_x d' + afd'') - \omega^2(add'' + bd'^2) + f_{tt} - aff_{xx} - bf_x^2 = 0. \tag{5}$$

In view of the fact that Eq. (5) is an ordinary differential equation with unknown function $\omega = \omega(f)$, we obtain

$$add'' + b(d')^2 = \beta(t)d, \tag{6}$$

$$adf_{xx} + 2bd'f_x + ad''f = \tilde{\gamma}(t)d. \tag{7}$$

We reduce the problem of determination of the solutions of Eq. (4) of the form (2) to the integration of a system of two ordinary differential equations one of which is linear. Equation (6) has the following particular solution:

$$d(x) = x^\alpha, \qquad a(t) = \frac{\alpha}{1-\alpha} b(t), \qquad \beta = 0, \quad \alpha \neq 1. \tag{8}$$

Substituting $d = x^\alpha$ in Eq. (7), we obtain

$$x^2 f_{xx} + 2(1-\alpha)xf_x + \alpha(\alpha - 1)f = \gamma(t)x^2, \tag{9}$$