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On exact solutions of nonlinear heat equation

Presented by Corresponding Member of the NAS of Ukraine A.G.Nikitin

A method for construction of exact solutions to nonlinear heat equation $u_t = (F(u)u_x)_x + G(u)u_x + H(u)$ which is based on ansatz $p(x) = \omega_1(t)\,\varphi(u) + \omega_2(t)$ is proposed. The function $p(x)$ here is a solution of equation $(p')^2 = Ap^2 + B$, and the functions $\omega_1(t)$, $\omega_2(t)$ and $\varphi(u)$ can be found from the condition that this ansatz reduces the nonlinear heat equation to a system of two ordinary differential equations with unknown functions $\omega_1(t)$ and $\omega_2(t)$.

Key words: group-theoretical methods, exact solutions, nonlinear heat equation, generalized variable separation

1. Introduction

The paper is devoted to construction of exact solutions to nonlinear heat equation

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left[F(u)\frac{\partial u}{\partial x}\right] + G(u)\frac{\partial u}{\partial x} + H(u). \qquad (1)$$

This equation under $G(u) \equiv const$ describes unsteady state heat transfer in a medium that is moving with a constant velocity, where the thermal conductivity coefficient and the reaction speed coefficient are arbitrary functions of temperature. The soliton solutions of equation (1) are presented in [1].

In the case $G(u) \equiv 0$ we have equation

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left[F(u)\frac{\partial u}{\partial x}\right] + H(u), \qquad (2)$$

which describes unsteady state heat transfer in an unmovable medium. Group classification of a class of equations of this type and exact solutions for different functions $F(u)$ and $H(u)$ are presented in [1–5].

In this paper we propose a method for constructing new exact solutions to equations (1) and (2). To solve these equations we use the ansatz

$$p(x) = \omega_1(t)\varphi(u) + \omega_2(t), \qquad (3)$$

which contains the unknown functions $\omega_1(t)$, $\omega_2(t)$ and $\varphi(u)$, whereas the function $p(x)$ is *a priori* predefined. Assume that $p(x)$ is a solution of equation

$$(p')^2 = Ap^2 + B,$$

and then determine functions $\omega_1(t)$, $\omega_2(t)$ and $\varphi(u)$ using the reduction idea. Namely, assume that the ansatz (3) reduces given equation to a system of two ordinary differential equations with unknown functions $\omega_1(t)$ and $\omega_2(t)$. This approach gives a description of a class of equations of the forms (1) and (2) that have solutions of the form (3), as well as an effective technique for constructing such solutions. An ansatz of the form (3) is used in [6,7] for constructing exact solutions of nonlinear wave equations and Korteweg–de Vries equations.

2. Exact solutions of equation (1)

In this section we determine the functions $F(u)$, $G(u)$ and $H(u)$, for which equation (1.1) has solutions of the form

$$x = \omega_1(t)\varphi(u) + \omega_2(t), \qquad (4)$$

i.e. admits the ansatz (4). This ansatz contains the three unknown functions $\omega_1(t)$, $\omega_2(t)$ and $\varphi(u)$. These functions will be determined from the condition that the ansatz (4) reduces equation (1) to a system of two ordinary differential equations with unknown functions $\omega_1(t)$ and $\omega_2(t)$. In order to obtain this system we substitute equation (4) into equation (1):

$$-\frac{\omega_1'}{\omega_1}\frac{\varphi}{\varphi'} - \frac{\omega_2'}{\omega_1}\frac{1}{\varphi'} = \left(-F\frac{\varphi''}{(\varphi')^3} + F'\frac{1}{(\varphi')^2}\right)\frac{1}{\omega_1^2} + \frac{1}{\omega_1}\frac{G}{\varphi'} + H(u), \qquad (5)$$

If there exists a solution of equation (1) of the form (4) than the obtained equation (2.2) means that the functions

$$\frac{\varphi}{\varphi'},\ \frac{1}{\varphi'},\ -F\frac{\varphi''}{(\varphi')^3} + F'\frac{1}{(\varphi')^2},\ \frac{G}{\varphi'},\ H \qquad (6)$$

are linearly dependent. The functions $\frac{\varphi}{\varphi'}$, $\frac{1}{\varphi'}$ are linearly independent, so all other functions (6) should obey the condition that they are representable as a linear combination of the functions $\frac{\varphi}{\varphi'}$, $\frac{1}{\varphi'}$. We have

$$-F\frac{\varphi''}{(\varphi')^3} + F'\frac{1}{(\varphi')^2} = \lambda_1\frac{\varphi}{\varphi'} + \mu_1\frac{1}{\varphi'}, \qquad (7)$$

$$G = \lambda_2 \varphi + \mu_2, \tag{8}$$

$$H = \lambda_3 \frac{\varphi}{\varphi'} + \mu_3 \frac{1}{\varphi'}, \tag{9}$$

for some $\lambda_i, \mu_i \in R$. Substitute (7)–(9) into equation (5):

$$\left(-\frac{\omega_1'}{\omega_1} - \frac{\lambda_1}{\omega_1^2} - \frac{\lambda_2}{\omega_1} - \lambda_3 \right) \frac{\varphi}{\varphi'} + \left(-\frac{\omega_2'}{\omega_1} - \frac{\mu_1}{\omega_1^2} - \frac{\mu_2}{\omega_1} - \mu_3 \right) \frac{1}{\varphi'} = 0. \tag{10}$$

The functions $\frac{\varphi}{\varphi'}$ and $\frac{1}{\varphi'}$ are linearly independent, so equation (10) splits into a system of equations

$$\frac{\omega_1'}{\omega_1} + \frac{\lambda_1}{\omega_1^2} + \frac{\lambda_2}{\omega_1} + \lambda_3 = 0, \tag{11}$$

$$\frac{\omega_2'}{\omega_1} + \frac{\mu_1}{\omega_1^2} + \frac{\mu_2}{\omega_1} + \mu_3 = 0. \tag{12}$$

Let $F'(u) \neq 0$. Integrating equation (7) which is linear with respect to the function $F = F(u)$, we find

$$F = \left(\lambda_1 \int \varphi du + \mu_1 u + A \right) \varphi', \tag{13}$$

where A is an arbitrary constant. As a result we can formulate the following theorem.

Theorem 1. Let $F'(u) \neq 0$ in equation (1). If equation (1) admits the ansatz (4), then the functions $F(u)$, $G(u)$ and $H(u)$ are defined by the formulas (13), (8) and (9), respectively, whereas $\omega_1(t)$ and $\omega_2(t)$ are solutions of the system of equations (11), (12).

In accordance with Theorem 1, the function $\varphi(u)$ in the ansatz (4) is arbitrary, whereas the functions $F(u)$, $G(u)$ and $H(u)$ can be represented via the function $\varphi(u)$. Finding solutions of form (4) of equation (1) is reduced to integrating the system of equations (11), (12). Rewrite this system in terms of new functions v_1 and v_2,

$$v_1 = \frac{1}{\omega_1}, \; v_2 = \frac{\omega_2}{\omega_1}.$$

Then this system transforms to

$$v_1' = \lambda_1 v_1^3 + \lambda_2 v_1^2 + \lambda_3 v_1, \tag{14}$$

$$v_2' = (\lambda_1 v_1^2 + \lambda_2 v_1 + \lambda_3) v_2 + \mu_1 v_1^2 + \mu_2 v_1 + \mu_3. \tag{15}$$

Consider three possible cases.

a) Case $\lambda_1 \neq 0$, $\lambda_2 = \lambda_3 = 0$. The equation (1) has the form

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left[F(u)\frac{\partial u}{\partial x}\right] + \mu_2 \frac{\partial u}{\partial x} + \mu_3 \frac{1}{\varphi'}, \qquad (16)$$

where the function $F(u)$ is defined by the formula (13). The general solution of the system (14), (15) for $\lambda_2 = \lambda_3 = 0$ is

$$v_1 = \left[-2\lambda_1(t + c_1)\right]^{-\frac{1}{2}},$$

$$v_2 = -\frac{\mu_1}{\lambda_1} + \mu_2 t\left[-2\lambda_1(t + c_1)\right]^{-\frac{1}{2}} - \frac{\mu_3}{3\lambda_1}\left[-2\lambda_1(t + c_1)\right] + c_2\left[-2\lambda_1(t + c_1)\right]^{-\frac{1}{2}},$$

where c_1 and c_2 are arbitrary constants. As a result we have the following solution to equation (16):

$$\varphi(u) = \left[-2\lambda_1(t + c_1)\right]^{-\frac{1}{2}} x + \frac{\mu_1}{\lambda_1} - \mu_2 t\left[-2\lambda_1(t + c_1)\right]^{-\frac{1}{2}} + \frac{\mu_3}{3\lambda_1}\left[-2\lambda_1(t + c_1)\right] - c_2\left[-2\lambda_1(t + c_1)\right]^{-\frac{1}{2}}. \qquad (17)$$

Setting $\mu_1 = \mu_2 = \mu_3 = 0$ and c_2=0 in (17) we obtain automodel solutions

$$\varphi(u) = \left[-2\lambda_1(t + c_1)\right]^{-\frac{1}{2}} x \qquad (18)$$

to equation

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left[F(u)\frac{\partial u}{\partial x}\right], \qquad (19)$$

where the function $F(u)$ is defined by the formula (13). Solutions of the form (18) are studied in [5].

b) Case $\lambda_1 = \lambda_2 = 0$, $\lambda_3 \neq 0$. The general solution of the system (14),(15) is defined by the formulas

$$v_1 = c_1 \exp(\lambda_3 t),$$

$$v_2 = \frac{\mu_1 c_1}{\lambda_3}\exp(2\lambda_3 t) + \mu_2 c_1 t \exp(\lambda_3 t) - \frac{\mu_3}{\lambda_3} + c_3 \exp(\lambda_3 t).$$

Equation (1) in this case becomes

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left[(\mu_1 u + A)\varphi' \frac{\partial u}{\partial x}\right] + \mu_2 \frac{\partial u}{\partial x} + \frac{1}{\varphi'}(\lambda_3 \varphi + \mu_3), \qquad (20)$$

and has the following family of solutions:

$$\varphi(u) = c_1 x \exp(\lambda_3 t) - \frac{\mu_1 c_1^2}{\lambda_3}\exp(2\lambda_3 t) - \mu_2 c_1 t \exp(\lambda_3 t) + \frac{\mu_3}{\lambda_3} - c_3 \exp(\lambda_3 t). \qquad (21)$$

When we set $\mu_2 = 0$ in (21) we obtain a family of solutions

$$\varphi(u) = c_1 x \exp(\lambda_3 t) - \frac{\mu_1 c_1^2}{\lambda_3}\exp(2\lambda_3 t) + \frac{\mu_3}{\lambda_3} - c_3 \exp(\lambda_3 t).$$

to equation

$$\frac{\partial u}{\partial t}=\frac{\partial}{\partial x}\left[(\mu_1 u+A)\varphi'\frac{\partial u}{\partial x}\right]+\frac{1}{\varphi'}(\lambda_3\varphi+\mu_3). \qquad (22)$$

c) Case $\lambda_1=0$, $\lambda_2\neq 0$, $\lambda_3\neq 0$. In this case equation (1.1) has the form

$$\frac{\partial u}{\partial t}=\frac{\partial}{\partial x}\left[(\mu_1 u+A)\varphi'\frac{\partial u}{\partial x}\right]+(\lambda_3\varphi+\mu_2)\frac{\partial u}{\partial x}+\frac{1}{\varphi'}(\lambda_3\varphi+\mu_3), \qquad (23)$$

and the general solution of the system (14), (15) is defined as

$$v_1=\frac{\lambda_3 c_1\exp(\lambda_3 t)}{c_2-\lambda_2 c_1\exp(\lambda_3 t)},$$

$$v_2=-\frac{\mu_1\lambda_3 c_1}{\lambda_2}\frac{\exp(\lambda_3 t)}{c_2-\lambda_2 c_1\exp(\lambda_3 t)}\cdot\ln\left|c_2-\lambda_2 c_1\exp(\lambda_3 t)\right|+(\mu_2\lambda_3-\mu_3\lambda_2)c_1 t\frac{\exp(\lambda_3 t)}{c_2-\lambda_2 c_1\exp(\lambda_3 t)}$$
$$-\frac{\mu_3 c_2}{\lambda_3\left(c_2-\lambda_2 c_1\exp(\lambda_3 t)\right)}+\frac{\lambda_3 c_3\exp(\lambda_3 t)}{c_2-\lambda_2 c_1\exp(\lambda_3 t)}.$$

Substituting these expressions for v_1 and v_2 into $\varphi(u) = v_2 x - v_2$ we obtain solutions to equation (23).

3. Exact solutions of equation (2)

In order to construct exact solutions of equation (2) we can use the substitution

$$p(x)=\omega_1(t)\varphi(u), \qquad (24)$$

where $p(x)$ is a solution of equation

$$(p')^2=Ap^2+B,\ \ A\neq 0,\ \ B\neq 0.$$

Determine the functions $\omega_1(t)$ and $\varphi(u)$ from the condition that the ansatz (24) reduces equation (2) to an ordinary differential equations with the unknown function $\omega_1(t)$. then we obtain the following system of equations for determining the functions $F(u)$, $G(u)$ and $H(u)$:

$$-F\frac{\varphi''}{(\varphi')^3}+F'\frac{1}{(\varphi')^2}=\lambda_2\frac{\varphi}{\varphi'}, \qquad (25)$$

$$-FA\frac{\varphi^2\varphi''}{(\varphi')^3}+F'A\frac{\varphi^2}{(\varphi')^2}+FA\frac{\varphi}{\varphi'}+H=\lambda_3\frac{\varphi}{\varphi'}, \qquad (26)$$

where $\lambda_2,\lambda_3\in R$. Suppose that $F'(u)\neq 0$. Then integrating equation (25) which is linear with respect to the function $F = F(u)$, we have

$$F=\left(\lambda_2\int\varphi du+c_1\right)\varphi', \qquad (27)$$

where c_1 is a constant. The function $\omega_1(t)$ can be determined from the equation

$$\frac{\omega_1'}{\omega_1} + \lambda_2 B \frac{1}{\omega_1^2} + \lambda_3 = 0. \qquad (28)$$

In the case $\lambda_3 \neq 0$, then the solution of (28) is the function

$$\omega_1^2 = \frac{c_2}{\lambda_3}\exp(-2\lambda_3 t) - \frac{\lambda_2}{\lambda_3}B,$$

where c_2 is a constant, and in the case $\lambda_3 = 0$ the solution is

$$\omega_1^2 = -2\lambda_2 Bt + c_2,$$

where c_2 is a constant, and $\lambda_2 \neq 0$.

From equations (25) and (26) we have

$$H = \frac{1}{\varphi'}\left(-\lambda_2 A\varphi^3 - AF\varphi + \lambda_3\varphi\right). \qquad (29)$$

As a result we formulate the next theorem.

Theorem 2. If equation (2) admits the ansatz (24) and $F'(u) = 0$, then the functions $F(u)$ and $H(u)$ are defined by the formulas (27) and (29), respectively, whereas the function $\omega_1(t)$ is a solution of equation (28).

Obtained solutions of equation (2) can be generalized by means of the substitutions

$$\varphi(u) = \omega_1(t)ch\left[k(x + c_3)\right] + \omega_2(t)sh\left[k(x + c_3)\right], \qquad (30)$$

if $A = k^2 > 0$, and

$$\varphi(u) = \omega_1(t)\cos\left[k(x + c_3)\right] + \omega_2(t)\sin\left[k(x + c_3)\right],$$

if $A = -k^2 < 0$.

For example, consider substitution (30). If the functions $F(u)$ and $H(u)$ are defined by the formulas (27) and (29), respectively, and $A = k^2 > 0$, then substitution (30) reduces equation (2) to a system

$$\omega_1' = \left(-\lambda_2 k^2\omega_1^2 + \lambda_2 k^2\omega_2^2\right)\omega_1 + \lambda_3\omega_1, \qquad (31)$$

$$\omega_2' = \left(-\lambda_2 k^2\omega_1^2 + \lambda_2 k^2\omega_2^2\right)\omega_2 + \lambda_3\omega_2, \qquad (32)$$

Let $\omega_1 \neq 0$. From equations (31),(32) we derive that $\omega_2 = c\omega_1$, c is a constant. Equation (31) rewrites as

$$\omega_1' == \lambda_2 k^2(c^2 - 1)\omega_1^3 + \lambda_3\omega_1. \qquad (33)$$

If $\lambda_3 \neq 0$, then the solution of equation (33) is

$$\omega_1^2 = \left[\frac{c_2}{\lambda_3} \exp(-2\lambda_3 t) - \frac{\lambda_2}{\lambda_3} k^2 (c^2 - 1) \right]^{-1},$$

where $c_2 \neq 0$ is a constant.

The solution of equation (2) is

$$\varphi(u) = \left[\frac{c_2}{\lambda_3} \exp(-2\lambda_3 t) - \frac{\lambda_2}{\lambda_3} k^2 (c^2 - 1) \right]^{-\frac{1}{2}} \left[ch[k(x + c_3)] + c \cdot sh[k(x + c_3)] \right].$$

If $\lambda_3 = 0$, then equation (33) solves as

$$\omega_1^2 = \left[-2\lambda_2 k^2 (c^2 - 1)t + c_2 \right]^{-1},$$

where c_2 is a constant and $\lambda_2 \neq 0$.

As a result, we have the following solution for equation (2):

$$\varphi(u) = \left[-2\lambda_2 k^2 (c^2 - 1)t + c_2 \right]^{-\frac{1}{2}} \left[ch[k(x + c_3)] + c \cdot sh[k(x + c_3)] \right]$$

The case $\omega_1 \neq 0$ reduces to integrating equation

$$\omega_2' = \lambda_2 k^2 \omega_2^3 + \lambda_3 \omega_2.$$

4. Conclusion

We have described equations of the form (1) that admit the ansatz (4). The functions $F(u)$, $G(u)$ and $H(u)$ in equation (1) can be represented in terms of $\varphi(u)$, and the corresponding system for finding $\omega_1(t)$ and $\omega_2(t)$ can be integrated. Voluntary choice of the function $\varphi(u)$ in the ansatz (4) allows one to find solutions of equation (1), that should satisfy predefined conditions. All this is true also for equation (2) which is a special case of equation (1). Moreover, the ansatz (24) gives essentially new solutions of equation (2).

The techniques of constructing solutions of equations (1) and (2) described in Sections 2 and 3 can also be efficiently applied for constructing solutions of plenty more equations, for example nonlinear wave equations [7].

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Про точні розв'язки нелінійного рівняння теплопровідності

Запропоновано метод побудови точних розв'язків нелінійного рівняння теплопровідності $u_t = (F(u)u_x)_x + G(u)u_x + H(u)$, який ґрунтується на використанні підстановки $p(x) = \omega_1(t)\,\varphi(u) + \omega_2(t)$, де функція $p(x)$ є розв'язком рівняння $(p')^2 = Ap^2 + B$, а функції $\omega_1(t)$, $\omega_2(t)$ та $\varphi(u)$ знаходяться з умови, що дана підстановка редукує рівняння до системи двох звичайних диференціальних рівнянь з невідомими функціями $\omega_1(t)$ та $\omega_2(t)$.

Ключові слова: теоретико-групові методи, інваріантні розв'язки, нелінійне рівняння теплопровідності, узагальнене розділення змінних.

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О точных решениях нелинейного уравнения теплопроводности

Предложен метод построения точных решений нелинейного уравнения теплопроводности $u_t = (F(u)u_x)_x + G(u)u_x + H(u)$, основанный на использовании подстановки $p(x) = \omega_1(t)\,\varphi(u) + \omega_2(t)$, где функция р (х) является решением уравнения $(p')^2 = Ap^2 + B$, а функции $\omega_1(t)$, $\omega_2(t)$ и $\varphi(u)$ находятся из условия, что данная подстановка редуцирует уравнение к системе двух обыкновенных дифференциальных уравнений с неизвестными функциями $\omega_1(t)$ и $\omega_2(t)$.

Ключевые слова: теоретико-групповые методы, точные решения, нелинейное уравнение теплопроводности, обобщённое разделение переменных.

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