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GENERALIZED SEPARATION OF VARIABLES AND EXACT SOLUTIONS OF NONLINEAR EQUATIONS

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We consider a generalized procedure of separation of variables in nonlinear hyperbolic-type equations and Korteweg–de-Vries-type equations. We construct a broad class of exact solutions of these equations that cannot be obtained by the Lie method and method of conditional symmetries.

1. Introduction

One of efficient methods for the construction of exact solutions of linear equations of mathematical physics is the method of separation of variables. For equations with two independent variables x and t and an unknown function u, this method is based on seeking exact solutions in the form of the product of functions of different arguments, i.e.,

$$u = a(x)\,b(t). \tag{1}$$

A method for the construction of exact solutions of nonlinear partial differential equations that generalizes the classical method of separation of variables was presented in [1]. Solutions are sought in the form of a finite sum of k terms, namely,

$$u(x,t) = \sum_{i=1}^{k} f_i(t)\,a_i(x), \tag{2}$$

where $f_i(t)$ and $a_i(x)$ are smooth functions to be determined. Exact solutions with generalized separation of variables that contain more than two terms were given in [2, 3].

Representation (1) can be regarded as an ansatz that reduces the equation under study to an ordinary differential equation with an unknown function $a = a(x)$ (or an unknown function $b = b(t)$). In [4], the following generalization of ansatz (1) for nonlinear equations was proposed:

$$u = \sum_{i=1}^{m} \omega_i(t)\,a_i(x) + f(x,t), \qquad m \geq 1. \tag{3}$$

Ansatz (3) contains an unknown function $f(x,t)$, m unknown functions $a_i(x)$, and m unknown functions $\omega_i(t)$, which are determined from the condition that ansatz (3) reduces the equation under study to a system of

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m ordinary differential equations with unknown functions $\omega_i(t)$. The determination of this system was illustrated by examples of nonlinear wave equations. If $m = 1$ in (3), then this system reduces to an ordinary differential equation with an unknown function $\omega_1(t)$.

In the present paper, we continue the investigations described in [4]. Parallel with (3), we consider the ansatz obtained from (3) by the substitution $u \mapsto x$, $x \mapsto u$, $t \mapsto t$, i.e.,

$$x = \sum_{i=1}^{m} \omega_i(t)\, a_i(u) + f(u,t). \tag{4}$$

Solutions (3) and (4) are called solutions with separated variables, and the method used for their construction is called the generalized procedure of separation of variables. Note that we do not require that the function $f(x,t)$ in ansatz (3) be representable in the form of the finite sum (2).

In the present paper, we use ansatzes of the type (3), (4) for the construction of exact solutions of the nonlinear hyperbolic-type equation

$$\frac{\partial^2 u}{\partial t^2} = au\frac{\partial^2 u}{\partial x^2} + b\left(\frac{\partial u}{\partial x}\right)^2 + c \tag{5}$$

and the Korteweg–de-Vries-type equation

$$\frac{\partial u}{\partial t} + F(u)\left(\frac{\partial u}{\partial x}\right)^k + \frac{\partial^3 u}{\partial x^3} = 0, \tag{6}$$

where k is a real parameter. In Eq. (5), we can always set $a = 1$ using a local transformation. Equation (5) was investigated in [5, 6]. Equation (6) was studied in [7] by the method of conditional symmetry. We also consider the following generalizations of Eqs. (5) (for $c = 0$) and (6):

$$\frac{\partial^2 u}{\partial t^2} = au\frac{\partial^2 u}{\partial x^2} + b\left(\frac{\partial u}{\partial x}\right)^2 + \Phi(t)u,$$

$$\frac{\partial u}{\partial t} + F(t,u)\left(\frac{\partial u}{\partial x}\right)^k + \frac{\partial^3 u}{\partial x^3} = 0.$$

Solutions of these equations given in the present paper cannot be obtained by methods of group analysis [8].

2. Separation of Variables for Eq. (5)

For the construction of exact solutions of Eq. (5), we use the ansatz

$$u = \omega(t) + f(x,t), \tag{7}$$

which is a special case of the general ansatz (3). This ansatz is a generalization of the substitution

$$u(x,t) = \varphi(x) + \psi(t),$$

which is often used for finding exact solutions of nonlinear equations in mathematical physics. Substituting (7) into (5), we obtain the equation

$$\omega'' + f_{tt} = a(\omega + f)f_{xx} + bf_x^2 + c, \tag{8}$$

which must be an ordinary differential equation with an unknown function $\omega = \omega(t)$. This implies that the coefficient af_{xx} of ω in Eq. (8) can be represented in the form $af_{xx} = 2a\mu_2(t)$. Therefore,

$$f = \mu_2(t)x^2 + \mu_1(t)x + \tilde{\mu}_0(t),$$

where $\tilde{\mu}_0(t)$, $\mu_1(t)$, and $\mu_2(t)$ are certain functions of t. Using (7), we obtain the ansatz

$$u = \mu_2(t)x^2 + \mu_1(t)x + \mu_0(t), \qquad \mu_0(t) = \tilde{\mu}_0(t) + \omega(t). \tag{9}$$

Substituting (9) into Eq. (5), we obtain the following system of equations for the functions $\mu_i(t)$:

$$\mu_2'' = (2a + 4b)\mu_2^2,$$

$$\mu_1'' = (2a + 4b)\mu_1\mu_2,$$

$$\mu_0'' = 2a\mu_0\mu_2 + b\mu_1^2 + c.$$

This system of equations was obtained by different methods in [5].

Consider the following special case of Eq. (5):

$$\frac{\partial^2 u}{\partial t^2} = au\frac{\partial^2 u}{\partial x^2} + b\left(\frac{\partial u}{\partial x}\right)^2. \tag{10}$$

To construct exact solutions of Eq. (10), we use the ansatz

$$u = \omega(t)d(x) + f(x,t), \tag{11}$$

which is a special case of the general ansatz (3). Ansatz (11) reduces Eq. (10) to the equation

$$\omega''d - \omega(af_{xx}d + 2bf_xd' + afd'') - \omega^2(add'' + bd'^2) + f_{tt} - aff_{xx} - bf_x^2 = 0. \tag{12}$$

Equation (12) must be an ordinary differential equation with an unknown function $\omega = \omega(t)$. Hence,

$$add'' + bd'^2 = \beta d, \qquad \beta \in \mathbb{R}, \tag{13}$$

$$adf_{xx} + 2bd'f_x + ad''f = \tilde{\gamma}(t)d. \tag{14}$$

As a result, the problem of finding exact solutions of the form (11) for Eq. (10) is reduced to the integration of a system of two ordinary differential equations one of which is nonlinear. Equation (13) has the following particular solution:

$$d = x^{\alpha}, \quad a = \frac{\alpha}{1-\alpha}b, \quad \beta = 0, \quad \alpha \neq 1. \tag{15}$$

Substituting $d = x^{\alpha}$ into Eq. (14), we get

$$x^2 f_{xx} + 2(1-\alpha)x f_x + \alpha(\alpha-1)f = \gamma(t)x^2, \tag{16}$$

where

$$\gamma(t) = \frac{1-\alpha}{b}\tilde{\gamma}(t).$$

Consider three cases.

Case 1: $\alpha = 2$. It follows from (15) that $a = -2b$. For $\alpha = 2$, the function

$$f = \gamma(t)x^2 \ln|x| + \tilde{\beta}(t)x^2 + \delta(t)x$$

is a general solution of Eq. (16). Using (11), we get

$$u = \gamma(t)x^2 \ln|x| + \beta(t)x^2 + \delta(t)x, \quad \beta(t) = \tilde{\beta}(t) + \omega(t). \tag{17}$$

Substituting (17) into Eq. (10), we establish that $\delta(t) = 0$ and obtain the following system of equations for the functions $\beta(t)$ and $\gamma(t)$:

$$\gamma'' = -2\beta\gamma^2, \quad \beta'' = -2b\beta\gamma + b\gamma^2. \tag{18}$$

System (18) can be solved completely in the implicit form. If $\gamma = \gamma(t)$ is a solution of the first equation of system (18), then the functions $\gamma(t)$ and $\frac{1}{\gamma}\int\frac{dt}{\gamma^2}$ form a fundamental system of solutions of the homogeneous equation $\beta'' = -2b\beta\gamma$ [6]. Therefore, the general solution of the second equation of system (18) can be determined (in quadratures) by using the solution of the first equation.

System (18) has the particular solution

$$\gamma = -\frac{3}{b}t^{-2}, \quad \beta = c_1 t^3 + c_2 t^{-2} - \frac{9}{5b}t^{-2}\ln|t|.$$

As a result, we obtain the following exact solution of Eq. (10) for $a = -2b$:

$$u = -\frac{3}{b}t^{-2}x^2 \ln|x| + \left(c_1 t^3 + c_2 t^{-2} - \frac{9}{5b} t^{-2} \ln|t| \right) x^2,$$

where c_1 and c_2 are arbitrary constants.

Case 2: $\alpha = 3$. It follows from (15) that $a = -\frac{3}{2}b$. For $\alpha = 3$, the function

$$f = -\gamma(t)x^2 \ln|x| + \beta(t)x^2 + \tilde{\delta}(t)x^3$$

is a general solution of Eq. (16). Using (11), we get

$$u = -\gamma(t)x^2 \ln|x| + \beta(t)x^2 + \delta(t)x^3, \qquad \delta(t) = \tilde{\delta}(t) + \omega(t). \tag{19}$$

Substituting (19) into Eq. (10), we establish that $\gamma(t) = 0$ and obtain the following system of equations for the functions $\beta(t)$ and $\delta(t)$:

$$\delta'' = 0, \qquad \beta'' = b\beta^2. \tag{20}$$

System (20) has the particular solution

$$\delta = c_1 t + c_2, \qquad \beta = \frac{6}{b}t^{-2}.$$

Therefore, the exact solution of Eq. (10) for $a = -\frac{3}{2}b$ is the function

$$u = \frac{6}{b}t^{-2}x^2 + (c_1 t + c_2)x^3,$$

where c_1 and c_2 are arbitrary constants.

The ansatz $u = \beta(t)x^2 + \delta(t)x^3$ is a special case of the more general ansatz

$$u = \mu_3(t)x^3 + \mu_2(t)x^2 + \mu_1(t)x + \mu_0(t). \tag{21}$$

Substituting ansatz (21) into Eq. (10), we obtain the following system of equations for the functions $\mu_i(t)$:

$$\mu_3'' = 0,$$

$$\mu_2'' = b\mu_2^2 - 3b\mu_1\mu_3,$$

$$\mu_1'' = b\mu_1\mu_2 - 9b\mu_0\mu_3,$$

$$\mu_0'' = -3b\mu_0\mu_2 + b\mu_1^2.$$

Note that ansatz (21) was considered in [9].

Case 3: $\alpha \neq 1, 2, 3$. The function

$$f = \frac{\gamma(t)}{(\alpha-2)(\alpha-3)}x^2 + \tilde{\mu}_2(t)x^\alpha + \mu_0(t)x^{-1+\alpha}$$

is a general solution of Eq. (16). Using (11), we get

$$u = \mu_1(t)x^2 + \mu_2(t)x^\alpha + \mu_0(t)x^{-1+\alpha}, \tag{22}$$

where

$$\mu_0(t), \quad \mu_1(t) = \frac{\gamma(t)}{(\alpha-2)(\alpha-3)}, \quad \text{and} \quad \mu_2(t) = \tilde{\mu}_2(t) + \omega(t)$$

are the unknown functions to be determined. Substituting (22) into Eq. (19), we establish that $\mu_0(t) = 0$ and obtain the following system of equations for the functions $\mu_1(t)$ and $\mu_2(t)$:

$$\mu_1'' = (2a+4b)\mu_1^2, \tag{23}$$

$$\mu_2'' = \left(-\frac{a^2b}{(a+b)^2} + \frac{2a^2+6ab}{a+b}\right)\mu_1\mu_2. \tag{24}$$

Equation (23) has the particular solution

$$\mu_1 = \frac{3}{a+2b}t^{-2}. \tag{25}$$

Substituting (25) into (24) and using (15), we obtain the following linear equation for the function $\mu_2(t)$:

$$t^2\mu_2'' = (9\alpha - 3\alpha^2)\mu_2. \tag{26}$$

Equation (26) has the following solutions:

$$\mu_2 = t^{1/2}[c_1\cos(\sigma\ln t) + c_2\sin(\sigma\ln t)] \quad \text{if} \quad \sigma^2 = 3\alpha^2 - 9\alpha - \frac{1}{4} > 0,$$

$$\mu_2 = t^{1/2}[c_1 t^{\sigma} + c_2 t^{-\sigma}] \quad \text{if} \quad \sigma^2 = \frac{1}{4} - 3\alpha^2 + 9\alpha > 0,$$

and

$$\mu_2 = t^{1/2}[c_1 + c_2 \ln t] \quad \text{if} \quad \alpha = \frac{9 \pm 2\sqrt{21}}{6},$$

where c_1 and c_2 are arbitrary constants.

Thus, Eq. (10) has the following exact solutions:

$$u = \frac{3}{a+2b} t^{-2} x^2 + t^{1/2} x^{\alpha} [c_1 \cos(\sigma \ln t) + c_2 \sin(\sigma \ln t)] \quad \text{if} \quad \sigma^2 = 3\alpha^2 - 9\alpha - \frac{1}{4} > 0,$$

$$u = \frac{3}{a+2b} t^{-2} x^2 + t^{1/2} x^{\alpha} [c_1 t^{\sigma} + c_2 t^{-\sigma}] \quad \text{if} \quad \sigma^2 = \frac{1}{4} - 3\alpha^2 + 9\alpha > 0,$$

and

$$u = \frac{3}{a+2b} t^{-2} x^2 + t^{1/2} x^{\alpha} [c_1 + c_2 \ln t] \quad \text{if} \quad \alpha = \frac{9 \pm 2\sqrt{21}}{6} > 0,$$

where c_1 and c_2 are arbitrary constants.

Note that the ansatz $u = \mu_1 x^2 + \mu_2 x^4$ is a special case of the more general ansatz

$$u = \mu_4(t)x^4 + \mu_3(t)x^3 + \mu_2(t)x^2 + \mu_1(t)x + \mu_0(t),$$

where the functions $\mu_i(t)$ satisfy the system of equations

$$\mu_4'' = -\frac{8}{3} b\mu_2\mu_4 + b\mu_3^2,$$

$$\mu_3'' = \frac{4}{3} b\mu_2\mu_3 - 8b\mu_1\mu_4,$$

$$\mu_2'' = \frac{4}{3} b\mu_2^2 - 2b\mu_1\mu_3 - 16b\mu_0\mu_4,$$

$$\mu_1'' = \frac{4}{3} b\mu_1\mu_2 - 8b\mu_0\mu_3,$$

$$\mu_0'' = -\frac{8}{3} b\mu_0\mu_2 + b\mu_1^2.$$

This ansatz was considered in [9].

The results obtained for Eq. (10) can be generalized to the equation

$$\frac{\partial^2 u}{\partial t^2} = au\frac{\partial^2 u}{\partial x^2} + b\left(\frac{\partial u}{\partial x}\right)^2 + \phi(t)u. \tag{27}$$

We have the following cases:

Case 1: $a = -2b$. The ansatz

$$u = \gamma(t)x^2 \ln|x| + \beta(t)x^2$$

reduces Eq. (27) to the system

$$\gamma'' = -2b\gamma^2 + \phi\gamma, \quad \beta'' = -2b\beta\gamma + \beta\gamma^2 + \phi\beta.$$

Case 2: $a = -\frac{3}{2}b$. The ansatz

$$u = \beta(t)x^2 + \delta(t)x^3$$

reduces Eq. (27) to the system

$$\delta'' = \phi\delta, \quad \beta'' = b\beta^2 + \phi\beta.$$

Case 3: $a \neq -b,\ -2b,\ -\frac{3}{2}b$. The ansatz

$$u = \mu_1(t)x^2 + \mu_2(t)x^\alpha,$$

where

$$\alpha = \frac{a}{a+b},$$

reduces Eq. (27) to the system

$$\mu_1'' = (2a+4b)\mu_1^2 + \phi(\mu_1),$$

$$\mu_2'' = \left(-\frac{a^2 b}{(a+b)^2} + \frac{2a^2+6ab}{a+b}\right)\mu_1\mu_2 + \phi(\mu_2).$$

3. Separation of Variables for Eq. (6)

Let us clarify for what functions $F(u)$ Eq. (6) admits an ansatz of the form

$$x = \omega_1(t)d(u) + \omega_2(t). \tag{28}$$

We determine the functions $\omega_1(t)$, $\omega_2(t)$, and $d(u)$ from the condition that ansatz (28) reduces Eq. (6) to a system of two ordinary differential equations with unknown functions $\omega_1 = \omega_1(t)$ and $\omega_2 = \omega_2(t)$. Substituting (28) into Eq. (6), we get

$$-\frac{\omega_1'}{\omega_1}\frac{d}{d'} - \frac{\omega_2'}{\omega_1}\frac{1}{d'} + \frac{1}{\omega_1^k}\frac{F(u)}{(d')^k} - \frac{1}{\omega_1^3}\frac{d'''}{(d')^4} + \frac{1}{\omega_1^3}\frac{3(d'')^2}{(d')^5} = 0. \tag{29}$$

The functions $\frac{d}{d'}$ and $\frac{1}{d'}$, which multiply $-\frac{\omega_1'}{\omega_1}$ and $-\frac{\omega_2'}{\omega_1}$ in Eq. (29), are linearly independent. Assume that $k \neq 3$ in Eq. (29). We require that the coefficients of the functions $\frac{1}{\omega_1^k}$ and $\frac{1}{\omega_1^3}$ be representable in the form of a linear combination over the field of real numbers of the functions $\frac{d}{d'}$ and $\frac{1}{d'}$. We get

$$-\frac{d'''}{(d')^4} + 3\frac{(d'')^2}{(d')^5} = \lambda\frac{d}{d'} + \mu\frac{1}{d'}, \qquad \lambda, \mu \in \mathbb{R},$$

or

$$-d'd''' + 3(d'')^2 = \lambda d(d')^4 + \mu(d')^4. \tag{30}$$

In view of (30), Eq. (29) yields

$$F(u) = [\omega_1'\omega_1^{k-1} - \lambda\omega_1^{k-3}]d(d')^{k-1} + [\omega_2'\omega_1^{k-1} - \mu\omega_1^{k-3}](d')^{k-1}.$$

Therefore,

$$\omega_1'\omega_1^{k-1} - \lambda\omega_1^{k-3} = \lambda_1, \qquad \omega_2'\omega_1^{k-1} - \mu\omega_1^{k-3} = \lambda_2,$$

where λ_1 and λ_2 are constants. Thus, in the case $k \neq 3$, Eq. (6) admits ansatz (28) if the function $F(u)$ has the form

$$F(u) = \lambda_1 d(d')^{k-1} + \lambda_2(d')^{k-1}, \tag{31}$$

where $d = d(u)$ is an arbitrary solution of Eq. (30). For $\lambda = 0$, Eq. (30) can be completely integrated. Consider the following special cases of Eq. (30):

(a) $d = \ln u$ if $\lambda = 0$ and $\mu = 1$;

(b) $d = u^{1/2}$ if $\lambda = \mu = 0$;

(c) $d = \arcsin u$ if $\lambda = 0$ and $\mu = -1$;

(d) $d = \operatorname{arcsinh} u$ if $\lambda = 0$ and $\mu = 1$.

In case (a), using (31) we get

$$F(u) = (\lambda_1 \ln u + \lambda_2) u^{1-k}, \tag{32}$$

and Eq. (6) has the following solutions:

$$u = \exp\left[-\frac{k(k\lambda_1)^{-3/k}}{k-2} t^{(k-3)/k} + ct^{-1/k} + k(k\lambda_1 t)^{-1/k} x - \frac{\lambda_2}{\lambda_1}\right], \quad k \neq 2,$$

$$u = \exp\left[-(2\lambda_1)^{-3/2} t^{-1/2} \ln t + ct^{-1/2} + (2\lambda_1 t)^{-1/2} x - \frac{\lambda_2}{\lambda_1}\right], \quad k = 2,$$

where c is an arbitrary constant. The first of these functions is also a solution of Eq. (6) for $k = 3$ (this case is considered below).

In case (b), we have

$$F(u) = \lambda_1' u^{(2-k)/2} + \lambda_2' u^{(1-k)/2}, \quad \lambda_1' = 2^{1-k}\lambda_1, \quad \lambda_2' = 2^{1-k}\lambda_2. \tag{33}$$

In case (c), we get

$$F(u) = (\lambda_1 \arcsin u + \lambda_2)(1 - u^2)^{(1-k)/2}. \tag{34}$$

In case (d), we obtain

$$F(u) = (\lambda_1 \operatorname{arcsinh} u + \lambda_2)(1 + u^2)^{(1-k)/2}. \tag{35}$$

Functions (32)–(35) and the corresponding solutions of Eq. (6) were obtained by the method of conditional symmetry in [7].

Let $k = 3$ in Eq. (6). Using Eq. (29), we get

$$F(u) = \omega_1' \omega_1^2 d (d')^2 + \omega_2' \omega_1^2 (d')^2 + \frac{d'''}{d'} - \frac{3(d'')^2}{(d')^2}.$$

By virtue of the linear independence of the functions $d(d')^2$ and $(d')^2$, we obtain

$$\omega_1{}'\omega_1^2 = \lambda_1, \qquad \omega_2{}'\omega_1^2 = \lambda_2, \tag{36}$$

where λ_1 and λ_2 are arbitrary constants.

Thus, for $k = 3$, Eq. (6) admits ansatz (28) if the function $F(u)$ has the form

$$F(u) = \lambda_1 d(d')^2 + \lambda_2 (d')^2 + \frac{d'''}{d'} - \frac{3(d'')^2}{(d')^2}, \tag{37}$$

where $d = d(u)$ is an arbitrary smooth function. For example, if $d = \exp u$, then, using (37), we get

$$F(u) = \lambda_1 \exp(3u) + \lambda_2 \exp(2u) - 2.$$

Therefore, by virtue of (28) and (36), the equation

$$\frac{\partial u}{\partial t} + [\lambda_1 \exp(3u) + \lambda_2 \exp(2u) - 2]\left(\frac{\partial u}{\partial x}\right)^3 + \frac{\partial^3 u}{\partial x^3} = 0$$

has the exact solution

$$u = \ln\left[\frac{x + c_1}{(3\lambda_1 t + c_2)^{1/3}} - \frac{\lambda_2}{\lambda_1}\right],$$

where c_1 and c_2 are arbitrary constants.

The results obtained for Eq. (6) can be generalized to the equation

$$\frac{\partial u}{\partial t} + F(t,u)\left(\frac{\partial u}{\partial x}\right)^k + \frac{\partial^3 u}{\partial x^3} = 0. \tag{38}$$

The following statement is true:

Theorem 1. *Equation (38) admits ansatz (28) if and only if one of the following conditions is satisfied:*

(i) the function $F(t,u)$ has the form

$$F(t,u) = f(t)\, d(d')^{k-1} + g(t)(d')^{k-1},$$

where $d = d(u)$ is an arbitrary solution of Eq. (30), $f(t)$ and $g(t)$ are functions of t, and $\omega_1 = \omega_1(t)$ and $\omega_2 = \omega_2(t)$ satisfy the system of equations

$$\omega_1'\omega_1^{k-1} - \lambda\omega_1^{k-3} = f(t), \qquad \omega_2'\omega_1^{k-1} - \mu\omega_1^{k-3} = g(t); \tag{39}$$

(ii) $k = 3$ *and the function* $F(t, u)$ *has the form*

$$F(t,u) = f(t)d(d')^2 + g(t)(d')^2 + \frac{d'''}{d'} - \frac{3(d'')^2}{(d')^2},$$

where $d = d(u)$ *is an arbitrary smooth function,* $f(t)$ *and* $g(t)$ *are functions of* t, *and* $\omega_1 = \omega_1(t)$ *and* $\omega_2 = \omega_2(t)$ *satisfy the system of equations*

$$\omega_1'\omega_1^2 = f(t), \qquad \omega_2'\omega_1^2 = g(t).$$

In condition (ii) of Theorem 1, we can assume that $d = d(u)$ is an arbitrary smooth function that is a solution of Eq. (30).

Consider the following special case of Eq. (30): $d = \ln u$ if $\lambda = 0$ and $\mu = 1$. System (39) takes the form

$$\omega_1'\omega_1^{k-1} = f(t), \qquad \omega_2'\omega_1^{k-1} - \omega_1^{k-3} = g(t),$$

and has the general solution

$$\omega_1 = \left[k\int f(t)\,dt + c_1\right]^{1/k}, \tag{40}$$

$$\omega_2 = \int\left[k\int f(t) + c_1\right]^{-2/k} dt + \int g(t)\left[k\int f(t) + c_1\right]^{(1-k)/k} dt + c_2, \tag{41}$$

where c_1 and c_2 are arbitrary constants.

In this case, we have

$$F(t,u) = \left[f(t)\ln u + g(t)\right]u^{1-k},$$

and the equation

$$\frac{\partial u}{\partial t} + \left[f(t)\ln u + g(t)\right]u^{1-k}\left(\frac{\partial u}{\partial x}\right)^k + \frac{\partial^3 u}{\partial x^3} = 0$$

has the solution

$$u = \exp\left[\frac{1}{\omega_1(t)}x - \frac{\omega_2(t)}{\omega_1(t)}\right],$$

where ω_1 and ω_2 are defined by (40) and (41).

By analogy, we show that if $d = u^{1/2}$ and $\lambda = \mu = 0$, then

$$F(t,u) = f_1(t)u^{(2-k)/2} + g_1(t)u^{(1-k)/2},$$

$$f_1(t) = 2^{1-k} f(t), \quad g_1(t) = 2^{1-k} g(t),$$

and the solution of Eq. (38) is the function

$$u = \left[\frac{1}{\omega_1(t)} x - \frac{\omega_2(t)}{\omega_1(t)} \right]^2.$$

In the case where $d = \arcsin u$, $\lambda = 0$, and $\mu = -1$, we get

$$F(t, u) = \left[f(t) \arcsin u + g(t) \right] (1 - u^2)^{(1-k)/2},$$

and the solution of Eq. (38) is the function

$$u = \sin \left[\frac{1}{\omega_1(t)} x - \frac{\omega_2(t)}{\omega_1(t)} \right].$$

In the case where $d = \operatorname{arcsinh} u$, $\lambda = 0$, and $\mu = 1$, we have

$$F(t, u) = \left[f(t) \operatorname{arc\,sinh} u + g(t) \right] (1 + u^2)^{(1-k)/2},$$

and the solution of Eq. (38) is the function

$$u = \sinh \left[\frac{1}{\omega_1(t)} x - \frac{\omega_2(t)}{\omega_1(t)} \right],$$

where ω_1 and ω_2 are defined by (40) and (41).

4. Conclusions

The generalized procedure of separation of variables developed in the present paper can be used for the determination of solutions of a broad class of nonlinear differential equations, in particular, nonlinear wave equations. Examples of these equations were considered in [4]. By using ansatz (4) and ansatzes obtained from it by a rearrangement of the variables u, x, and t, one can construct solutions that cannot be obtained by methods of group analysis. The solutions of Eq. (5) presented in Sec. 2 have the form (2) with 2, 3, 4, and 5 terms, and, therefore, it is not efficient to seek them directly in the form of sum (2).

Ansatz (3) can be applied, as a rule, to equations with polynomial nonlinearity. If the construction of a solution in the form (3) is impossible, then we first seek a transformation that reduces the equation under study to equations with polynomial nonlinearity and then construct a solution of the latter in the form (3). In many cases, these transformations can be found if an equation admits an ansatz of the type (4).

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