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GENERALIZED PROCEDURE OF SEPARATION OF VARIABLES AND REDUCTION OF NONLINEAR WAVE EQUATIONS

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We propose a generalized procedure of separation of variables for nonlinear wave equations and construct broad classes of exact solutions of these equations that cannot be obtained by the classical Lie method and the method of conditional symmetries.

1. Introduction

One of efficient methods for the construction of exact solutions of linear partial differential equations is the method of separation of variables. For example, it is known that the heat equation

$$u_t - u_{xx} = 0 \tag{1}$$

has the solutions with separated variables [1]

$$u(x,t) = ke^{\lambda t}\cosh\left(\sqrt{\lambda}\,x+\delta\right), \qquad \lambda > 0,$$

$$u(x,t) = ke^{\lambda t}\cos\left(\sqrt{|\lambda|}\,x+\delta\right), \qquad \lambda < 0,$$

which can be obtained by the substitution

$$u = a(x)\,b(t). \tag{2}$$

A method for the construction of exact solutions of nonlinear differential equations that generalizes the classical method of separation of variables was presented in [2]. According to this method, solutions are sought in the form

$$u(x,t) = \sum_{i=1}^{k} f_i(t)\,a_i(x), \tag{3}$$

where $f_i(t)$ and $a_i(x)$ are smooth functions to be determined. In [3, 4], the construction of solutions in the form of this finite sum was used for the analysis of various classes of nonlinear equations.

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Substitution (2) can be regarded as an ansatz that reduces Eq. (1) to an ordinary differential equation with unknown function $a = a(x)$ (or with unknown function $b = b(t)$). Ansatz (2) is also efficient for the determination of particular solutions of nonlinear partial differential equations. In the present paper, we consider the following generalization of ansatz (2):

$$u = \sum_{i=1}^{m} \omega_i(t)\, a_i(x) + f(x,t), \qquad m \geq 1. \tag{4}$$

Ansatz (4) contains an unknown function $f(x,t)$, m unknown functions $a_i(x)$, and m unknown functions $\omega_i(t)$, which are determined from the condition that ansatz (4) reduces the equation considered to a system of m ordinary differential equations with unknown functions $\omega_i(t)$. We consider the problem of finding this system by using the following nonlinear wave equations as an example:

$$u_{tt} = uu_{xx} + au_x^2 + u^2, \tag{5}$$

$$u_{tt} = uu_{xx} + au_x^2 + b_1 x + b_2 u. \tag{6}$$

Solutions of the form (4) are called solutions with separated variables, and the method for the construction of these solutions is called the generalized procedure of separation of variables.

Note that we do not require that the function $f(x,t)$ in ansatz (4) be represented in the form of the finite sum (3). Equation (5) was investigated in [5]. For $b_1 = b_2 = 0$ and $a = 1$, Eq. (6) reduces to the equation $u_{tt} = uu_{xx} + au_x^2$, which was investigated in [5–8].

2. Separation of Variables for the Nonlinear Equation (5)

Consider the generalized procedure of separation of variables for the nonlinear equation (5). For the construction of solutions of this equation, we use the ansatz

$$u = \omega_1(t)\, d(x) + \omega_0(t), \tag{7}$$

where $\omega_0(t)$, $\omega_1(t)$, and $d(x)$ are unknown functions and, furthermore, the function $d(x)$ is not a constant. We determine these functions from the condition that ansatz (7) reduces Eq. (5) to a system of two ordinary differential equations with unknown functions $\omega_0(t)$ and $\omega_1(t)$. To obtain the required system of equations, we substitute (7) into (5). As a result, we get

$$\omega_1'' d + \omega_0'' - \omega_0^2 - \omega_1^2(dd'' + ad'^2 + d^2) - \omega_0\omega_1(d'' + 2d) = 0. \tag{8}$$

In Eq. (8), the coefficients 1 and d of the functions ω_0'' and ω_1'' are linearly independent over R. We require that the coefficients of the functions ω_1^2 and $\omega_0\omega_1$ be representable in the form of a linear combination of the functions 1 and d over R. This requirement means that Eq. (8) can be rewritten in the form

$$\phi_1(\omega_0, \omega_1, \omega_0'', \omega_1'') + d\phi_2(\omega_0, \omega_1, \omega_0'', \omega_1'') = 0, \tag{9}$$

where the functions ϕ_1 and ϕ_2 depend only on the variable t, and the function d depends only on the variable x. As a result, we obtain Eq. (9) with separated variables t and x. Thus, Eq. (9) splits into the system of two ordinary differential equations

$$\phi_1 = 0, \qquad \phi_2 = 0.$$

It follows from Eq. (8) that

$$d'' = \lambda + \mu d, \qquad \lambda, \mu \in R. \tag{10}$$

If $\mu \neq 0$ in (10), then performing the substitution

$$\hat{d} = \frac{\lambda}{\mu} + d, \qquad \hat{\omega}_0 = \omega_0 - \frac{\lambda}{\mu}\omega_1, \tag{11}$$

we obtain the ansatz

$$u = \omega_1(t)\,\hat{d}(x) + \hat{\omega}_0(t).$$

Substitution (10) does not change the function u but transforms $d(x)$ into the function $\hat{d}(x)$, for which $\hat{d}'' = \mu\hat{d}$. Thus, in ansatz (7), we can assume that

$$\hat{d}'' = \mu d, \quad \mu \neq 0. \tag{12}$$

Consider possible cases.

1. *Case* $\mu < 0$. A general solution of Eq. (12) has the form

$$d = C_1 \cos\left(\sqrt{|\mu|}x\right) + C_2 \sin\left(\sqrt{|\mu|}x\right), \tag{13}$$

where C_1 and C_2 are arbitrary constants and $C_1^2 + C_2^2 \neq 0$. Then

$$dd'' + ad'^2 + d^2 = A_1 \cos^2\left(\sqrt{|\mu|}x\right) + A_2 \sin^2\left(\sqrt{|\mu|}x\right) + A_3 \cos\left(\sqrt{|\mu|}x\right)\sin\left(\sqrt{|\mu|}x\right), \tag{14}$$

$$A_1 = -|\mu|C_1^2 + a|\mu|C_2^2 + C_1^2, \qquad A_2 = -|\mu|C_2^2 + a|\mu|C_1^2 + C_2^2,$$
$$A_3 = \left(-2|\mu| - 2a|\mu| + 2\right)C_1C_2. \tag{15}$$

Since the coefficient $dd'' + ad'^2 + d^2$ is a linear combination of the functions 1 and d over R, it follows from (14) and (15) that

$$|\mu| = \frac{1}{a+1}, \qquad a > -1.$$

Thus, ansatz (7) has the form

$$u = \left[C_1 \cos\left(\sqrt{\frac{1}{a+1}}x\right) + C_2 \sin\left(\sqrt{\frac{1}{a+1}}x\right)\right]\omega_1(t) + \omega_0(t)$$

and reduces Eq. (5) to the equation

$$\left[C_1 \cos\left(\sqrt{\frac{1}{a+1}}x\right) + C_2 \sin\left(\sqrt{\frac{1}{a+1}}x\right)\right]\omega_1'' + \omega_0'' - \omega_0^2 - \frac{2}{a+1}(C_1^2 + C_2^2)\,\omega_1^2$$

$$- \frac{2a+1}{a+1}\left[C_1 \cos\left(\sqrt{\frac{1}{a+1}}x\right) + C_2 \sin\left(\sqrt{\frac{1}{a+1}}x\right)\right]\omega_0\omega_1 = 0,$$

which splits into the system

$$\omega_0'' - \omega_0^2 - \frac{a}{a+1}(C_1^2 + C_2^2)\,\omega_1^2 = 0, \tag{16}$$

$$\omega_1'' - \frac{2a+1}{a+1}\omega_0\omega_1 = 0. \tag{17}$$

In the case $a = 1$, the obtained ansatz can be reduced to the ansatz

$$u = \cos\left(\frac{\sqrt{2}}{2}x\right)\omega_1(t) + \omega_0(t),$$

which was considered in [5].

If $a \neq -1/2$, then, using Eq. (17), we get

$$\omega_0 = \frac{a+1}{2a+1}\frac{\omega_1''}{\omega_1}. \tag{18}$$

Substituting (18) into Eq. (16), we obtain the following equation for the function ω_1:

$$\omega_1^2\omega_1^{\mathrm{IV}} - 2\omega_1\omega_1'''\omega_1' - \frac{3a+2}{2a+1}\omega_1(\omega_1'')^2 + 2\omega_1''(\omega_1')^2 - \frac{a(2a+1)}{(a+1)^2}(C_1^2 + C_2^2)\,\omega_1^5 = 0. \tag{19}$$

It can be shown that any solution of the equation

$$\omega_1'' = \frac{\lambda(2a+1)}{a+1}\omega_1^2, \qquad \lambda^2 = C_1^2 + C_2^2, \tag{20}$$

satisfies Eq. (19). If condition (20) is satisfied, then, using (18), we obtain $\omega_0 = \lambda\omega_1$. The corresponding solution of Eq. (5) has the form

$$u = \left[C_1 \cos\left(\sqrt{\frac{1}{a+1}}x\right) + C_2 \sin\left(\sqrt{\frac{1}{a+1}}x\right) + \varepsilon\sqrt{C_1^2 + C_2^2}\right]\omega_1(t), \quad \varepsilon = \pm 1,$$

where ω_1 is a solution of Eq. (20).

If

$$\lambda = \frac{6(a+1)}{2a+1},$$

then we obtain the following solution of Eq. (5):

$$u = \left[C_1 \cos\left(\sqrt{\frac{1}{a+1}}x\right) + C_2 \sin\left(\sqrt{\frac{1}{a+1}}x\right) + \frac{6(a+1)}{2a+1}\right]\rho(t),$$

where $\rho(t)$ is the Weierstrass function with invariants $g_2 = 0$, g_3, and

$$C_1^2 + C_2^2 = \frac{36(a+1)^2}{(2a+1)^2}.$$

2. *Case* $\mu > 0$. As in the previous case, it can be shown that ansatz (7) has the form

$$u = \left[C_1 \cosh\left(\sqrt{-\frac{1}{a+1}}x\right) + C_2 \sinh\left(\sqrt{-\frac{1}{a+1}}x\right)\right]\omega_1(t) + \omega_0(t), \qquad a < -1,$$

and reduces Eq. (5) to the system

$$\omega_0'' - \omega_0^2 - \frac{a}{a+1}(C_1^2 - C_2^2)\,\omega_1^2 = 0, \tag{21}$$

$$\omega_1'' - \frac{2a+1}{a+1}\omega_0\omega_1 = 0. \tag{22}$$

The system of equations (21), (22) is analogous to system (16), (17). Consequently, Eq. (5) has the solution

$$u = \left[C_1 \cosh\left(\sqrt{-\frac{1}{a+1}}x\right) + C_2 \sinh\left(\sqrt{-\frac{1}{a+1}}x\right) + \frac{6(a+1)}{2a+1}\right]\rho(t), \qquad a < -1,$$

where $\rho(t)$ is the Weierstrass function with invariants $g_2 = 0$, g_3, and

$$C_1^2 - C_2^2 = \frac{36(a+1)^2}{(2a+1)^2}.$$

Using the ansatz

$$u = \omega_2(t)\,d_2(x) + \omega_1(t)\,d_1(x) + \omega_0(t),$$

where the functions 1, $d_1(x)$, and $d_2(x)$ are linearly independent, for the construction of solutions of Eq. (5) and repeating the arguments presented above, we obtain the following ansatzes and reduced systems:

(a) the ansatz

$$u = \omega_2(t)\cos\left(\sqrt{\frac{1}{a+1}}x\right) + \omega_1(t)\sin\left(\sqrt{\frac{1}{a+1}}x\right) + \omega_0(t), \qquad a > -1,$$

and the reduced system

$$\omega_2'' - \frac{2a+1}{a+1}\omega_0\omega_2 = 0,$$

$$\omega_1'' - \frac{2a+1}{a+1}\omega_0\omega_1 = 0,$$

$$\omega_0'' - \omega_0^2 - \frac{a}{a+1}(\omega_1^2 + \omega_2^2) = 0;$$

(b) the ansatz

$$u = \omega_2(t)\cosh\left(\sqrt{-\frac{1}{a+1}}x\right) + \omega_1(t)\sinh\left(\sqrt{-\frac{1}{a+1}}x\right) + \omega_0(t), \qquad a < -1,$$

and the reduced system

$$\omega_2'' - \frac{2a+1}{a+1}\omega_0\omega_2 = 0,$$

$$\omega_1'' - \frac{2a+1}{a+1}\omega_0\omega_1 = 0,$$

$$\omega_0'' - \omega_0^2 - \frac{a}{a+1}(\omega_1^2 - \omega_2^2) = 0.$$

3. Separation of Variables for the Nonlinear Equation (6)

For the construction of Eq. (6), we first use the simple ansatz

$$u = f(t,x) + \omega, \tag{23}$$

which is a special case of the general ansatz (4). Ansatz (23) reduces (6) to the equation

$$\omega'' - \omega(f_{xx} + b_2) + f_{tt} - ff_{xx} - af_x^2 - b_1 f_x - b_2 f = 0, \tag{24}$$

which is an ordinary differential equation with unknown function $\omega = \omega(t)$. This implies that the coefficient $f_{xx} + b_2$ of ω in Eq. (24) can be represented in the form $f_{xx} + b_2 = 2\mu_2(t)$. Therefore,

$$f = \mu_2(t)x^2 + \mu_1(t)x + \tilde{\mu}_0(t),$$

where $\tilde{\mu}_0(t)$, $\mu_1(t)$, and $\mu_2(t)$ are certain functions of t. Using (23), we obtain the ansatz

$$u = \mu_2(t)x^2 + \mu_1(t)x + \mu_0(t), \quad \mu_0 = \tilde{\mu}_0 + \omega. \tag{25}$$

Substituting (25) into Eq. (6), we obtain the following system of equations for the functions $\mu_i(t)$:

$$\mu_2'' = (2+4a)\mu_2^2 + b_2\mu_2,$$

$$\mu_1'' = (2+4a)\mu_1\mu_2 + 2b_1\mu_2 + b_2\mu_1,$$

$$\mu_0'' = 2\mu_0\mu_2 + a\mu_1^2 + b_1\mu_1 + b_2\mu_0.$$

Ansatz (23) is a special case of the more general ansatz

$$u = f(t,x) + \omega(t)d(x), \tag{26}$$

which reduces Eq. (6) to the equation

$$\omega'' d - \omega(f_{xx}d + 2af_x d' + fd'' + b_1 d' + b_2 d) - \omega^2(dd'' - ad'^2) + f_{tt} - ff_{xx} - af_x^2 - b_1 f_x - b_2 f = 0. \tag{27}$$

Equation (27) is an ordinary differential equation with an unknown function $\omega = \omega(t)$. This yields

$$dd'' + ad'^2 = \beta d, \quad \beta \in R, \tag{28}$$

$$f_{xx}d + 2af_x d' + fd'' + b_1 d' + b_2 d = \alpha(t)d. \tag{29}$$

Therefore, the problem of finding exact solutions of the form (26) for Eq. (6) is reduced to the integration of a system of ordinary differential equations one of which is linear. Consider the two cases that correspond to the following particular solutions of Eq. (28):

$$d = x^3 \quad \text{if} \quad a = -\frac{2}{3}, \quad \beta = 0; \qquad d = x^4 \quad \text{if} \quad a = -\frac{3}{4}, \quad \beta = 0.$$

(a_1) *Case* $d = x^3$, $a = -\frac{2}{3}$. Substituting $d = x^3$ into Eq. (29), we obtain

$$x^2 f_{xx} - 4xf_x + 6f = (\alpha(t) - b_2)x^2 - b_1 x. \tag{30}$$

If $\alpha(t) = b_2$ and $b_1 = 0$, then the general solution of Eq. (30) has the form

$$f = \tilde{\mu}_3(t)x^3 + \mu_2(t)x^2.$$

With regard for (26), we get

$$u = \mu_3(t)x^3 + \mu_2(t)x^2, \qquad \mu_3 = \tilde{\mu}_3 + \omega. \tag{31}$$

Ansatz (31) is a special case of the more general ansatz

$$u = \mu_3(t)x^3 + \mu_2(t)x^2 + \mu_1(t)x + \mu_0(t). \tag{32}$$

Substituting (32) into Eq. (6) (the parameters b_1 and b_2 are arbitrary), we obtain the following system of equations for the functions $\mu_i(t)$:

$$\mu_3'' = b_2\mu_3,$$

$$\mu_2'' = -\frac{2}{3}\mu_2^2 + 2\mu_1\mu_3 + 3b_1\mu_3 + b_2\mu_2,$$

$$\mu_1'' = -\frac{2}{3}\mu_1\mu_2 + 6\mu_0\mu_3 + 2b_1\mu_2 + b_2\mu_1,$$

$$\mu_0'' = 2\mu_0\mu_2 - \frac{2}{3}\mu_1^2 + b_1\mu_1 + b_2\mu_0.$$

(a$_2$) *Case* $d = x^4$, $a = -\dfrac{3}{4}$. As in the previous case, we obtain the ansatz

$$u = \mu_4(t)x^4 + \mu_3(t)x^3 + \mu_2(t)x^2 + \mu_1(t)x + \mu_0(t), \tag{33}$$

where the functions $\mu_i(t)$ satisfy the system of equations

$$\mu_4'' = 2\mu_2\mu_4 - \frac{3}{4}\mu_3^2 + b_2\mu_4,$$

$$\mu_3'' = -6\mu_1\mu_2 - \mu_2\mu_3 + 4b_1\mu_4 + b_2\mu_3,$$

$$\mu_2'' = -\mu_2^2 - \frac{9}{2}\mu_1\mu_3 + 12\mu_0\mu_4 + 3b_1\mu_3 + b_2\mu_2,$$

$$\mu_1'' = 6\mu_0\mu_3 - 3\mu_1\mu_2 + 2b_1\mu_2 + b_2\mu_1,$$

$$\mu_0'' = 2\mu_0\mu_2 - \frac{3}{4}\mu_1^2 + b_1\mu_1 + b_2\mu_0.$$

For the construction of exact solutions of Eq. (6), we use the ansatz

$$u = f(t,x) + \omega_1(t)d(x) + \omega_0(t), \tag{24}$$

which contains four unknown functions $\omega_0(t)$, $\omega_1(t)$, $d(x)$, and $f(t,x)$, where the function $d(x)$ is not a constant. We determine these functions from the condition that ansatz (34) reduces Eq. (6) to a system of two ordinary differential equations with unknown functions $\omega_0(t)$ and $\omega_1(t)$. We obtain the required system of equations by analogy with Eq. (5). Substituting (34) into (6), we get

$$\omega_1'' + \omega_0'' - \omega_1(f_{xx}d + 2af_xd' + fd'' + b_1d' + b_2d) - \omega_1^2(dd'' + ad'^2)$$

$$- \omega_0\omega_1d'' - \omega_0(f_{xx} + b_2) + f_{tt} - ff_{xx} - af_x^2 - b_1f_x - b_2f = 0. \tag{35}$$

The coefficients 1 and d of the functions ω_0'' and ω_1'' in Eq. (35) are linearly independent over R. We require that the coefficients of the functions ω_1, ω_1^2, $\omega_0\omega_1$, and ω_0 and the function $f_{tt} - ff_{xx} - af_x^2 - b_1f_x - b_2f$ be representable in the form of a linear combination of the functions 1 and d whose coefficients are functions of t. It follows from Eq. (35) that

$$d'' = \lambda + \mu d, \quad \lambda, \mu \in R. \tag{10}$$

If $\mu = 0$, then one can easily verify that the function $u = u(t, x)$ has the form (10). If $\mu \neq 0$, then we can assume that

$$d'' = \mu d, \quad \lambda, \mu \in R. \tag{36}$$

Two cases are possible.

1. *Case* $\mu < 0$. Equation (29) has the general solution

$$d = C_1 \cos\left(\sqrt{|\mu|}x\right) + C_2 \sin\left(\sqrt{|\mu|}x\right),$$

where C_1 and C_2 are arbitrary constants and $C_1^2 + C_2^2 \neq 0$. Then

$$\begin{gathered} dd'' + ad'^2 = A_1 \cos^2\left(\sqrt{|\mu|}x\right) + A_2 \sin^2\left(\sqrt{|\mu|}x\right) + A_3 \cos\left(\sqrt{|\mu|}x\right)\sin\left(\sqrt{|\mu|}x\right), \\ A_1 = -|\mu|\left(C_1^2 - aC_2^2\right), \quad A_2 = -|\mu|\left(aC_1^2 + C_2^2\right), \\ A_3 = -2|\mu|(C_1C_2 + aC_1C_2). \end{gathered} \tag{37}$$

Since the coefficient $dd'' + ad'^2$ is a linear combination of the functions 1 and d over R, it follows from (37) that $a = -1$. The coefficient $f_{xx} + b_2$ of ω_0 in Eq. (35) can be represented in the form

$$f_{xx} + b_2 = \alpha_0(t) + \alpha_1(t)\, d\,,\ \alpha_0(t) + \alpha_1(t)\, d\,.$$

Therefore,

$$f = \mu_2(t)\, x^2 + \mu_1(t)\, x + \mu_0(t) - \frac{\alpha_1(t)}{|\mu|} d$$

for certain functions $\mu_i(t)$. Performing the substitution

$$\hat{\omega}_1(t) = \omega_1(t) - \frac{\alpha_1(t)}{|\mu|}, \qquad \hat{f} = f + \frac{\alpha_1(t)}{|\mu|} d$$

in ansatz (34), we obtain the ansatz

$$u = \hat{f} + \hat{\omega}_1(t)\, d + \omega_0(t).$$

Therefore, we can assume that, in ansatz (34),

$$f = \mu_2(t)\, x^2 + \mu_1(t)\, x + \mu_0(t).$$

Taking into account that the coefficient of ω_1 in Eq. (35) has the form

$$\beta_0(t) + \beta_1(t)\, d\,,$$

where $\beta_0(t)$ and $\beta_1(t)$ are certain functions of t, we get $\mu_1 = 0$ and $\mu_2 = 0$. Therefore, the substitution $\hat{\omega}_0 = \omega_0 + \mu_0$ reduces ansatz (34) to the ansatz

$$u = \omega_1(t)\, d(x) + \hat{\omega}_0(t)\,.$$

Therefore, we can assume that $f = 0$ in ansatz (34). As a result, we obtain the ansatz

$$u = \omega_1(t)\left[C_1 \cos\left(\sqrt{|\mu|}\,x\right) + C_2 \sin\left(\sqrt{|\mu|}\,x\right)\right] + \omega_0(t),$$

which reduces Eq. (6) with parameter $a = -1$ to the system

$$\omega_0'' - b_2\omega_0 - \mu(C_1^2 + C_2^2)\,\omega_1^2 = 0,$$

$$\omega_1'' - b_2\omega_1 - \mu\omega_0\omega_1 = 0,$$

provided that $b_1 = 0$.

2. *Case* $\mu > 0$. Equation (36) has the general solution

$$d = C_1 \cosh\left(\sqrt{|\mu|}\,x\right) + C_2 \sinh\left(\sqrt{|\mu|}\,x\right),$$

where C_1 and C_2 are arbitrary constants and $C_1^2 + C_2^2 \neq 0$. As in case 1, we show that $a = -1$ and ansatz (34) has the form

$$u = \omega_1(t)\left[C_1 \cosh\left(\sqrt{|\mu|}\,x\right) + C_2 \sinh\left(\sqrt{|\mu|}\,x\right)\right] + \omega_0(t)$$

and reduces Eq. (6) with parameter $a = -1$ to the system

$$\omega_0'' - b_2\omega_0 - \mu(C_1^2 - C_2^2)\,\omega_1^2 = 0,$$

$$\omega_1'' - b_2\omega_1 - \mu\omega_0\omega_1 = 0,$$

provided that $b_1 = 0$.

Note that, using ansatz (34), one can reduce Eq. (6) with parameter $a = -1$ to a system of two ordinary differential equations if $b_1 = 0$. If $b_1 \neq 0$ in Eq. (6), then we can use the ansatz

$$u = \omega_2(t)\, d_2(x) + \omega_1(t)\, d_1(x) + \omega_0(t),$$

where the functions 1, $d_1(x)$, and $d_2(x)$ are linearly independent. Reasoning as above, we obtain the following ansatzes and reduced systems for Eq. (6) with parameter $a = -1$:

(a) the ansatz

$$u = \omega_2(t)\sin\left(\sqrt{|\mu|}x\right) + \omega_1(t)\cos\left(\sqrt{|\mu|}x\right) + \omega_0(t), \qquad \mu < 0,$$

and the reduced system

$$\omega_0'' - b_2\omega_0 - \mu\omega_1^2 + \mu\omega_2^2 = 0,$$

$$\omega_1'' - b_2\omega_1 - b_1\sqrt{\mu}\,\omega_2 - \mu\omega_0\omega_1 = 0,$$

$$\omega_2'' - b_2\omega_2 - b_1\sqrt{\mu}\,\omega_2 - \mu\omega_0\omega_2 = 0;$$

(b) the ansatz

$$u = \omega_2(t)\sinh\left(\sqrt{|\mu|}x\right) + \omega_1(t)\cosh\left(\sqrt{|\mu|}x\right) + \omega_0(t), \qquad \mu > 0,$$

and the reduced system

$$\omega_0'' - b_2\omega_0 - \mu\omega_1^2 + \mu\omega_2^2 = 0,$$

$$\omega_1'' - b_2\omega_1 - b_1\sqrt{\mu}\,\omega_2 - \mu\omega_0\omega_1 = 0,$$

$$\omega_2'' - b_2\omega_2 - b_1\sqrt{\mu}\,\omega_1 - \mu\omega_0\omega_2 = 0.$$

4. Reduction and Exact Solutions of the Equation $u_{tt} = uu_{xx} + u_x^2$

The equation

$$u_{tt} = uu_{xx} + u_x^2 \tag{38}$$

is a special case of Eq. (6). Its group properties were studied in detail by the Lie method in [7, 8]. The algebra of invariance of Eq. (38) is four-dimensional and is generated by the operators

$$P_t = \frac{\partial}{\partial t}, \quad P_x = \frac{\partial}{\partial x}, \quad X = t\frac{\partial}{\partial t} - x\frac{\partial}{\partial x} - 4u\frac{\partial}{\partial u},$$
$$Y = -t\frac{\partial}{\partial t} + 2u\frac{\partial}{\partial u}. \tag{39}$$

Algebra (39) has only the following one-dimensional subalgebras:

$$\langle P_t \rangle, \ \langle P_x \rangle, \ \langle X+Y \rangle, \ \langle X+aY \rangle, \quad a \in R, \ a \neq 1; \quad \langle P_t + a(X+Y) \rangle, \quad a \in R.$$

The subalgebras different from $\langle P_t \rangle$ and $\langle P_x \rangle$ are associated with the following symmetric ansatzes:

(i) the subalgebra $\langle P_t + a(X+Y) \rangle$, $a \in R$, is associated with the symmetric ansatz

$$u = e^{-2at}\omega(z), \quad z = e^{at}x;$$

(ii) the subalgebra $\langle X+Y \rangle$ is associated with the symmetric ansatz

$$u = x^2\omega(t);$$

(iii) the subalgebra $\langle X+aY \rangle$, $a \in R$, $a \neq 1$, is associated with the symmetric ansatz

$$u = t^{\frac{2(a-2)}{1-a}}\omega(z), \quad z = e^{\frac{1}{1-a}}x.$$

The subalgebras $\langle P_t \rangle$, $\langle P_x \rangle$, and $\langle X+Y \rangle$ are associated with solutions of Eq. (38) with separated variables t and x. The other symmetric ansatzes for Eq. (38) are exhausted by ansatzes of the form

$$u = d(t)\omega(z), \quad z = a(t)x, \tag{40}$$

where $a(t)$ and $d(t)$ are certain functions of t. Generalizing ansatz (40), we pose the problem of finding ansatzes of the form

$$u = d(t)\omega(z) + f(t,x), \quad z = a(t)x + b(t) \tag{41}$$

that reduce Eq. (38) to ordinary differential equations with an unknown function $\omega = \omega(z)$. Ansatz (41) reduces Eq. (38) to the equation

$$B_0\omega'' - a^2d^2(\omega')^2 + B_1\omega' - a^2d^2\omega\omega'' + B_2\omega + B_3 = 0, \tag{42}$$

where

$$B_0 = d(a'x+b')^2 - a^2 df, \quad B_1 = d(a''x+b'') + 2d'(a'x+b') - 2adf_x,$$

$$B_2 = d'' - f_{xx}d, \quad B_3 = f_{tt} - f_x^2 - ff_{xx}.$$

If Eq. (42) is an ordinary differential equation with an unknown function $\omega = \omega(z)$, then the coefficient B_0 can be rewritten in the form

$$d'(a'x+b')^2 - a^2 df = \mu(z)\, a^2 d^2.$$

Hence,

$$f = -\mu(z)\, d + \frac{(a'x+b')^2}{a^2}.$$

Performing the substitution

$$\hat{\omega} = \omega - \mu(z), \quad \hat{f} = f + \mu(z)\, d,$$

we obtain the ansatz

$$u = d(t)\hat{\omega}(z) + \hat{f}(t,x), \quad z = a(t)x + b(t).$$

Therefore, we can assume that, in ansatz (41),

$$f = \frac{(a'x+b')^2}{a^2},$$

i.e., ansatz (41) has the form

$$u = d(t)\omega(z) + \frac{z_t^2}{z_x^2}, \quad z = a(t)x + b(t). \tag{43}$$

To determine ansatzes of the form (43), it is necessary to determine the following three functions: $a = a(t)$, $b = b(t)$, and $d = d(t)$. We solve this problem under the assumption that $d = a^2$. If $a'' = 0$, then we obtain the following ansatzes and reduced equations:

(a) the ansatz

$$u = \omega(z) + t^2, \quad z = x + \frac{1}{2}t^2$$

and the reduced equation

$$\omega\omega'' + (\omega')^2 - \omega' - 2 = 0;$$

(b) the ansatz

$$u = t^2\omega(z) + t^{-2}x^2 + 2\mu t^3 x + \mu^2 t^8, \quad z = tx + \frac{1}{6}\mu t^6, \quad \mu \in R,$$

and the reduced equation

$$\omega\omega'' + (\omega')^2 - 5\mu\omega' - 50\mu^2 = 0.$$

Assume that $a'' \neq 0$ in ansatz (43). Taking into account that $d = a^2$, we rewrite the reduced equation (42) in the form

$$-a^6\omega\omega'' - a^6(\omega')^2 + B_1\omega' + B_2\omega + B_3 = 0, \tag{44}$$

where

$$B_1 = a^2(a''x + b''), \quad B_2 = 2aa'',$$

$$B_3 = \frac{2(a''x+b'')^2}{a^2} + \frac{2(a'x+b')(a'''x+b''')}{a^2} - \frac{8a'(a'x+b')(a''x+b'')}{a^3} - \frac{2a''(a'x+b')^2}{a^3}.$$

If Eq. (44) is an ordinary differential equation with an unknown function $\omega = \omega(z)$, then the coefficient B_2 can be represented in the form $B_2 = 2\lambda a^6$, $\lambda \in R$, i.e., $a'' = \lambda a^5$. By analogy, we get

$$B_1 = \mu a^6(ax + b + \delta), \quad \mu, \delta \in R,$$

i.e.,

$$a^2(a''x + b'') = \mu a^6(ax + b + \delta). \tag{45}$$

It follows from (45) that $\mu = \lambda$ and

$$b'' = \lambda a^4(b + \delta), \tag{46}$$

$$a'''x + b''' = 4\lambda a^3 a'(ax + b + \delta) + \lambda a^4(a'x + b'). \tag{47}$$

Taking (46) and (47) into account, we determine the coefficient B_3:

$$B_3 = 2\lambda^2 a^6 (ax + b + \delta)^2 .$$

Thus,

$$B_1 = \lambda a^6 (z + \delta), \qquad B_2 = 2\lambda a^6, \qquad B_3 = 2\lambda^2 a^6 (z + \delta)^2, \tag{48}$$

and the reduced equation (44) has the form

$$\omega\omega'' + (\omega')^2 - \lambda (z + \delta)\omega' - 2\lambda\omega - 2\lambda^2 (z + \delta)^2 = 0. \tag{49}$$

We can assume that $\delta = 0$ in (49) and ansatz (43) has the form

$$u = a(t)^2 \omega(z) + \frac{(a'x + b')^2}{a^2}, \qquad z = a(t)x + b(t), \tag{50}$$

where $a = a(t)$ is a solution of the equation

$$a'' = \lambda a^5 \tag{51}$$

and $b = b(t)$ is a solution of the equation

$$b'' = \lambda a^4 b . \tag{52}$$

Ansatz (50) reduces Eq. (38) to the equation

$$\omega\omega'' + (\omega')^2 - \lambda z\omega' - 2\lambda\omega - 2\lambda^2 z^2 = 0.$$

System (51), (52) has the following particular solutions:

$$a = t^{-1/2}, \quad b = \mu t^{3/2}, \quad \mu \in R, \quad \text{if} \quad \lambda = \frac{3}{4},$$

$$a = \rho(t)^{-1/2}, \quad b = \frac{1}{3}\mu g_3^{-1} \rho(t)^{-1/2} \int_0^t \rho(s)\, ds, \quad \mu \in R, \quad \text{if} \quad \lambda = -\frac{3}{4} g_3,$$

where $\rho(t)$ is the Weierstrass function with invariants $g_2 = 0$ and g_3.

These solutions are associated with the following ansatzes and reduced equations:

(c) the ansatz

$$u = t^{-1}\omega(z) + \left(-\frac{1}{2}t^{-1}x + \frac{3}{2}\mu t\right)^2, \qquad z = t^{-1/2}x + \mu t^{3/2}, \quad \mu \in R,$$

and the reduced equation

$$\omega\omega'' + (\omega')^2 - \frac{3}{4}z\omega' - \frac{3}{2}\omega - \frac{9}{8}z^2 = 0;$$

(d) the ansatz

$$u = \rho(t)^{-1}\omega(z) + \rho(t)^{-1}\left[\frac{1}{2}z\rho(t)^{-1} + \frac{1}{3}\mu g_3^{-1}\rho(t)^{3/2}\right]^2,$$

$$z = \rho(t)^{-1/2}x + \frac{1}{3}\mu g_3^{-1}\rho(t)^{-1/2}\int_0^t \rho(s)\,ds$$

and the reduced equation

$$\omega\omega'' + (\omega')^2 + \frac{3}{4}g_3 z\omega' + \frac{3}{2}g_3\omega - \frac{9}{8}g_3^2 z^2 = 0.$$

For Eq. (38), ansatzes (a)–(d) can also be obtained by using conditional symmetries.

We present other ansatzes for Eq. (38). The ansatz $u = \omega(t)$ reduces Eq. (38) to the equation $\omega'' = 0$. As a result, we obtain the following family of solutions of Eq. (38):

$$u = C_1 t + C_2, \qquad C_1 \text{ and } C_2 \text{ are constants.}$$

The ansatz $u = \omega(t)$ is a special case of the more general ansatz

$$u = f(t,x) + \omega(t), \tag{53}$$

which reduces Eq. (38) to the equation

$$\omega'' - f_{xx}\omega + f_{tt} - f_x^2 - ff_{xx} = 0,$$

which is an ordinary differential equation with an unknown function $u = \omega(t)$. Hence, $f_{xx} = 2\mu_2(t)$. Therefore,

$$f = \mu_2(t)x^2 + \mu_1(t)x + \tilde{\mu}_0(t),$$

where $\tilde{\mu}_0(t)$, $\mu_1(t)$, and $\mu_2(t)$ are certain functions of t. Using (53), we obtain the ansatz

$$u = \mu_2(t)x^2 + \mu_1(t)x + \mu_0(t), \qquad \mu_0 = \tilde{\mu}_0 + \omega. \tag{54}$$

Ansatz (54) was presented in [6]. Function (54) is a solution of Eq. (38) if the functions $\mu_i(t)$ satisfy the system of equations

$$\mu_2'' = 6\mu_2^2, \qquad \mu_1'' = 6\mu_2\mu_1, \qquad \mu_0'' = \mu_1^2 + 2\mu_2\mu_0,$$

i.e., we obtain the system presented in Sec. 3 for $a = 1$. Therefore, Eq. (38) has the solutions

$$u = x^2\rho(t) + \Lambda(t),$$

where $\Lambda(t)$ is the Lamé function, which satisfies the equation $\Lambda'' = 2\rho\Lambda$,

$$u = t^{-2}x^{-2} + C_1 t^3 x + \frac{C_1^2}{54}t^8 + C_2 t^{-1} + C_3 t^2,$$

where C_1, C_2, and C_3 are arbitrary constants, and

$$u = 2(x + t^2)$$

if $\mu_2(t) = 0$, $\mu_1(t) = 2$, and $\mu_0(t) = 2t^2$ in (54).

Ansatz (54) is a special case of the more general ansatz

$$u = f(t, x) + \omega(t)d(x). \tag{55}$$

Ansatz (55) reduces Eq. (38) to the equation

$$\omega''d - \omega^2(d'^2 + dd'') - \omega(2f_x d' + fd'' + f_{xx}d) + f_{tt} - f_x^2 - ff_{xx} = 0,$$

which must be an ordinary differential equation with an unknown function $\omega = \omega(t)$. This yields

$$d'^2 + dd'' = \beta d, \qquad \beta \in R, \tag{56}$$

$$f_{xx}d + 2f_x d' + fd'' = \alpha(t)d. \tag{57}$$

Equation (56) has the following particular solutions:

$$d = x^{1/2} \quad \text{if} \quad \beta = 0 \quad \text{and} \quad d = x^2 \quad \text{if} \quad \beta = 6.$$

Substituting $d = x^{1/2}$ into Eq. (57), we obtain

$$x^2 f_{xx} + x f_x - \frac{1}{4}f = \alpha(t)\,x^{1/2}. \tag{58}$$

Equation (58) has the general solution

$$f = \frac{4}{15}\alpha(t)\,x^2 + \mu_1(t)\,x^{1/2} + \mu_0(t)\,x^{-1/2}.$$

Using (55), we obtain

$$u = \mu_2(t)\,x^2 + \mu_1(t)\,x^{1/2} + \mu_0(t)\,x^{-1/2}, \tag{59}$$

where $\mu_2(t) = \frac{4}{15}\alpha(t) + \omega(t)$, $\mu_1(t)$, and $\mu_0(t)$ are unknown functions to be determined. Substituting (59) into (38), we get $\mu_0 = 0$, $\mu_2'' = 6\mu_2^2$, and $\mu_1'' = \frac{15}{4}\mu_1\mu_2$. Therefore, Eq. (38) has the following solutions:

$$u = t^{-2}x^2 + (C_1 t^{5/2} + C_2 t^{-3/2})\,x^{1/2}, \quad u = t\,(C_1 x + C_2)^{1/2},$$

where C_1 and C_2 are arbitrary constants.

For $d = x^2$, we do not obtain new ansatzes.

5. Conclusions

The generalized procedure of separation of variables developed in the present paper can be used for the determination of solutions of a broad class of nonlinear differential equations and systems of nonlinear reaction-diffusion equations (see, e.g., [9]). By using ansatz (4), one can construct solutions that cannot be obtained using the classical Lie method or the method of conditional symmetries. For example, ansatz (32) for Eq. (6) with parameter $a = -2/3$ and ansatz (33) with parameter $a = -3/4$ determine exactly these classes of solutions of Eq. (6). These solutions have the form (3) with four and five terms, respectively, and, hence, it is inefficient to seek these solutions in the form of sum (3).

Ansatz (53), which is a special case of the general ansatz (4), is simple. By using this ansatz, one can easily obtain ansatz (54) presented in [6] for Eq. (38). This ansatz can also be used for the determination of solutions of other equations, e.g., the nonlinear equations of acoustics $u_{tt} = uu_{xx}$ and the Boussinesq equation $u_{tt} + u_x^2 + uu_{xx} + u_{xxxx} = 0$.

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