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ON HADAMARD COMPOSITIONS OF ENTIRE DIRICHLET SERIES AND DIRICHLET SERIES ABSOLUTELY CONVERGING IN HALF-PLANE

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For an entire Dirichlet series $F(s) = \sum\limits_{k=0}^{\infty} f_k \exp\{s\lambda_k\}$ and a Dirichlet series $G(s) = \sum\limits_{k=0}^{\infty} g_k \exp\{s\lambda_k\}$ with finite abscissa of the absolute convergence the Dirichlet series $(F{*}G)(s) = \sum\limits_{k=0}^{\infty} f_k g_k \exp\{s\lambda_k\}$ is called the *Hadamard composition.* In terms of generalized orders the growth of this composition and their derivatives is investigated. A relation between the behavior of the maximal terms of the Hadamard composition of the derivatives and of the derivative of the Hadamard composition is established.

Key words: Dirichlet series, Hadamard composition, generalized order, maximal term.

1. Introduction

For power series $f(z) = \sum\limits_{k=0}^{\infty} f_k z^k$ and $g(z) = \sum\limits_{k=0}^{\infty} g_k z^k$ with the convergence radii $R[f]$ and $R[g]$ the series $(f * g)(z) = \sum\limits_{k=0}^{\infty} f_k g_k z^k$ is called the Hadamard composition. It is well known [1, 2] that $R[f * g] \ge R[f]R[g]$. Properties of this composition obtained by

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J. Hadamard find applications [2, 3] in the theory of analytic continuation of the functions represented by power series. We remark also that singular points of the Hadamard composition are investigated in the article [4].

For $0 \le r < R[f]$ let $\mu_f(r) = \max\{|f_k|r^k\colon k \ge 0\}$ be the maximal term of the power expansion of f. Studying [5, 6] a connection between the growth of maximal terms of a derivative of the Hadamard's composition of two entire functions f and g and the Hadamard composition of their derivatives M. Sen [6], in particular proved, that if the function $(f*g)$ has order ϱ and lower order λ then for every $\varepsilon > 0$ and all $r \ge r_0(\varepsilon)$

$$r^{(n+2)\lambda-1-\varepsilon} \le \frac{\mu_{f^{(n+1)}*g^{(n+1)}}(r)}{\mu_{(f*g)^{(n)}}(r)} \le r^{(n+2)\varrho-1+\varepsilon}.$$

Since Dirichlet series with positive increasing to $+\infty$ exponents are direct generalizations of power series, a problem becomes natural on similar results for a Hadamard composition of such series.

So, let $\Lambda = (\lambda_k)$ be an increasing to $+\infty$ sequence of nonnegative numbers ($\lambda_0 = 0$), and $S(\Lambda, A)$ be a class of Dirichlet series

$$F(s) = \sum_{k=0}^{\infty} f_k \exp\{s\lambda_k\}, \quad s = \sigma + it \tag{1}$$

with the exponents Λ and the abscissa of absolute convergence $\sigma_a[F] = A$. If $F \in (\Lambda, A_1)$ and $G(s) = \sum\limits_{k=0}^{\infty} g_k \exp\{s\lambda_k\} \in (\Lambda, A_2)$ the Dirichlet series

$$(F*G)(s) = \sum_{k=0}^{\infty} f_k g_k \exp\{s\lambda_k\} \tag{2}$$

is called [7] the *Hadamard composition* of F and G.

For a Dirichlet series (1) with $\sigma_a[F] = A[F] = A > -\infty$ and $\sigma < A$ we put $M(\sigma, F) = \sup\{|F(\sigma+it)|: t \in \mathbb{R}\}$, and let $\mu(\sigma, F) = \max\{|f_k|\exp\{\sigma\lambda_k\}: k \ge 0\}$ be the maximal term, $\nu(\sigma, F) = \max\{k\colon |f_k|\exp\{\sigma\lambda_k\} = \mu(\sigma, F)\}$ be the central index and $\Lambda(\sigma, F) = \lambda_{\nu(\sigma,F)}$. The following result is proved in [7].

Proposition 1. *Let $n \in \mathbb{Z}_+$, $m \in \mathbb{N}$ and $m > n$. If $\sigma_a[F] = \sigma_a[G] = +\infty$ and* $\ln k = o(\lambda_k \ln \lambda_k)$ *as $k \to \infty$ then*

$$\varlimsup_{\sigma\to+\infty} \frac{1}{\sigma} \ln \frac{\mu(\sigma, (F*G)^{(m)})}{\mu(\sigma, (F*G)^{(n)})} = (m-n)\varrho_R[f*G]$$

and (if $\varrho_R[f*G] < +\infty$)

$$\varliminf_{\sigma\to+\infty} \frac{1}{\sigma} \ln \frac{\mu(\sigma, (F*G)^{(m)})}{\mu(\sigma, (F*G)^{(n)})} = (m-n)\lambda_R[f*G],$$

where $\varrho_R[f]$ and $\lambda_R[f]$ are respectively the R-order and the lower R-order of entire Dirichlet series (1). *If $\sigma_a[F] = \sigma_a[G] = 0$ and* $\ln k = o(\lambda_k/\ln \lambda_k)$ *as $k \to \infty$ then*

$$\varlimsup_{\sigma\uparrow 0} |\sigma| \ln \frac{\mu(\sigma, (F*G)^{(m)})}{\mu(\sigma, (F*G)^{(n)})} = (m-n)\varrho^{(0)}[f*G]$$

and

$$\varliminf_{\sigma\uparrow 0}|\sigma|\ln\frac{\mu(\sigma,(F*G)^{(m)})}{\mu(\sigma,(F*G)^{(n)})}=(m-n)\lambda^{(0)}[f*G],$$

where $\varrho^{(0)}[f]$ *and* $\lambda^{(0)}[f]$ *are respectively the order and the lower order of Dirichlet series* (1) *with* $\sigma_a[F]=0$.

Here we will consider the case, when $\sigma_a[F]=+\infty$ and $\sigma_a[G]\in(-\infty+\infty)$.

2. Convergence and growth

We put

$$A[F]=\varliminf_{k\to+\infty}\frac{1}{\lambda_k}\ln\frac{1}{|f_k|},\quad \overline{A}[F]=\varlimsup_{k\to+\infty}\frac{1}{\lambda_k}\ln\frac{1}{|f_k|}.$$

It is known ([8],[9]) that $\sigma_a[F]\le A[F]$ and if $\ln k=o(\lambda_k)$ as $k\to\infty$ then $\sigma_a[F]=A[F]$. It is easy to see that if $A[F]>-\infty$ and $A[G]>-\infty$ then $A[F*G]\ge A[F]+A[G]$. Therefore, if $\sigma_a[F]=+\infty$ and $A[G]>-\infty$ then $A[F*G]=+\infty$.

We remark also [7] that if $\sigma_a[F]=+\infty$ and $\sigma_a[G]>-\infty$ then

$$\sigma_a[F*G]\ge\sigma_a[F]+\sigma_a[G]=+\infty.$$

Further, we will also assume that $\sigma_a[G]=A[G]$.

In [7] it is proved that

$$\sigma_a[F*G]=\sigma_a[(F*G)^{(n)}]=\sigma_a[F^{(n)}*G^{(n)}]$$

for every $n\in\mathbb{N}$, whence we get the following statement.

Proposition 2. *If* $\sigma_a[F]=+\infty$ *and* $\sigma_a[G]>-\infty$ *then*

$$\sigma_a[F*G]=\sigma_a[(F*G)^{(n)}]=\sigma_a[F^{(n)}*G^{(n)}]=\sigma_a[F]=+\infty$$

for every $n\in\mathbb{N}$.

By L we denote the class of non-negative continuous on $(-\infty,+\infty)$ functions α such that $\alpha(x)=\alpha(x_0)\ge 0$ for $x\le x_0$ and $\alpha(x)\uparrow+\infty$ as $x_0\le x\to+\infty$. We say that $\alpha\in L^0$, if $\alpha\in L$ and $\alpha((1+o(1))x)=(1+o(1))\alpha(x)$ as $x\to+\infty$. Finally, $\alpha\in L_{si}$, if $\alpha\in L$ and $\alpha(cx)=(1+o(1))\alpha(x)$ as $x\to+\infty$ for each $c\in(0,+\infty)$, i.e. α is a slowly increasing function. Clearly, $L_{si}\subset L^0$.

If $\alpha\in L$, $\beta\in L$ and $F\in(\Lambda,+\infty)$ then the quantities

$$\varrho_{\alpha,\beta}[F]:=\varlimsup_{\sigma\to+\infty}\frac{\alpha(\ln M(\sigma,F))}{\beta(\sigma)},\quad \lambda_{\alpha,\beta}[F]:=\varliminf_{\sigma\to+\infty}\frac{\alpha(\ln M(\sigma,F))}{\beta(\sigma)}\tag{3}$$

are called the generalized (α,β)-order and the generalized lower (α,β)-order of F. If in (3) we substitute $\ln\mu(\sigma,F)$ instead of $\ln M(\sigma,F)$ then we obtain quantities, which we denote by $\varrho_{\alpha,\beta}[\ln\mu,F]$ and $\lambda_{\alpha,\beta}[\ln\mu,F]$ respectively. Substituting $\Lambda(\sigma,F)$ instead of $\ln M(\sigma,F)$ by analogy we define $\varrho_{\alpha,\beta}[\Lambda,F]$ and $\lambda_{\alpha,\beta}[\Lambda,F]$. The following lemma is true [9, 10].

Lemma 1. *Let* $\alpha \in L_{si}$, $\beta \in L^0$ *and* $\dfrac{d\beta^{-1}(c\alpha(x))}{d\ln x} = O(1)$ *as* $x \to +\infty$ *for each* $c \in (0, +\infty)$ *and* $F \in (\Lambda, +\infty)$. *If for each* $c \in (0, +\infty)$

$$\ln k = o(\lambda_k \beta^{-1}(c\alpha(\lambda_k))), \quad k \to \infty, \tag{4}$$

then

$$\varrho_{\alpha,\beta}[F] = \varrho_{\alpha,\beta}[\ln \mu, F] = \overline{\lim_{k\to\infty}} \frac{\alpha(\lambda_k)}{\beta\left(\dfrac{1}{\lambda_k} \ln \dfrac{1}{|f_k|}\right)}. \tag{5}$$

If, moreover, $\alpha(\lambda_{k+1}) \sim \alpha(\lambda_k)$ *and* $\varkappa_k[F] := \dfrac{\ln |f_k| - \ln |f_{k+1}|}{\lambda_{k+1} - \lambda_k} \nearrow +\infty$ *as* $k_0 \le k \to \infty$ *then*

$$\varrho_{\alpha,\beta}[F] = \varrho_{\alpha,\beta}[\ln \mu, F] = \underline{\lim_{k\to\infty}} \frac{\alpha(\lambda_k)}{\beta\left(\dfrac{1}{\lambda_k} \ln \dfrac{1}{|f_k|}\right)}. \tag{6}$$

We need also the following lemmas.

Lemma 2. *If* $F \in (\Lambda, +\infty)$, $\alpha(e^x) \in L_{si}$, $\beta \in L^0$ *and* $\alpha(x) = o(\beta(x))$ *as* $x \to +\infty$ *then* $\varrho_{\alpha,\beta}[\ln \mu, F] = \varrho_{\alpha,\beta}[\Lambda, F]$ *and* $\lambda_{\alpha,\beta}[\ln \mu, F] = \lambda_{\alpha,\beta}[\Lambda, F]$.

Proof. We use the equality (see [8], [9])

$$\ln \mu(\sigma, F) - \ln \mu(0, F) = \int_0^\sigma \Lambda(x)dx, \quad 0 \le \sigma < +\infty. \tag{7}$$

From (7) it follows that for every $\varepsilon > 0$ and all $\sigma \ge 0$

$$\frac{\varepsilon\sigma}{1+\varepsilon}\Lambda\left(\frac{\sigma}{1+\varepsilon}, F\right) \le \ln \mu(\sigma, F) - \ln \mu(0, F) \le \sigma\Lambda(\sigma, F). \tag{8}$$

Hence $\ln \mu(\sigma, F) \ge \Lambda\left(\dfrac{\sigma}{1+\varepsilon}, F\right)$ for all $\sigma > 0$ large enough and, thus,

$$\varrho_{\alpha,\beta}[\Lambda, F] = \overline{\lim_{\sigma\to+\infty}} \frac{\alpha(\Lambda(\sigma, F))}{\beta(\sigma)} \le \overline{\lim_{\sigma\to+\infty}} \frac{\alpha(\mu((1+\varepsilon)\sigma, F))}{\beta((1+\varepsilon)\sigma)} \overline{\lim_{\sigma\to+\infty}} \frac{\beta((1+\varepsilon)\sigma)}{\beta(\sigma)},$$

$$\lambda_{\alpha,\beta}[\Lambda, F] = \underline{\lim_{\sigma\to+\infty}} \frac{\alpha(\Lambda(\sigma, F))}{\beta(\sigma)} \le \underline{\lim_{\sigma\to+\infty}} \frac{\alpha(\mu((1+\varepsilon)\sigma, F))}{\beta((1+\varepsilon)\sigma)} \overline{\lim_{\sigma\to+\infty}} \frac{\beta((1+\varepsilon)\sigma)}{\beta(\sigma)}.$$

Therefore, $\varrho_{\alpha,\beta}[\Lambda, F] \le \varrho_{\alpha,\beta}[\ln \mu, F]B(\varepsilon)$ and $\lambda_{\alpha,\beta}[\Lambda, F] \le \lambda_{\alpha,\beta}[\ln \mu, F]B(\varepsilon)$, where in view of condition $\beta \in L^0$ we get [11] $B(\varepsilon) = \overline{\lim_{\sigma\to+\infty}} \dfrac{\beta((1+\varepsilon)\sigma)}{\beta(\sigma)} \to 1$ as $\varepsilon \to 0$, and thus, $\varrho_{\alpha,\beta}[\Lambda, F] \le \varrho_{\alpha,\beta}[\ln \mu, F]$ and $\lambda_{\alpha,\beta}[\Lambda, F] \le \lambda_{\alpha,\beta}[\ln \mu, F]$.

On the other hand, if on the contrary $\varrho_{\alpha,\beta}[\Lambda, F] < \varrho_{\alpha,\beta}[\ln \mu, F]$ then for every $\varrho \in (\varrho_{\alpha,\beta}[\Lambda, F], \varrho_{\alpha,\beta}[\ln \mu, F])$ and all $\sigma \ge \sigma_0(\varrho)$ we have $\Lambda(\sigma, F) \le \alpha^{-1}(\varrho\beta(\sigma))$ and, thus, $\ln\,\mu(\sigma,F)\le(1+o(1))\sigma\alpha^{-1}(\varrho\beta(\sigma))$ as $\sigma\to+\infty$, i.e.

$$\begin{aligned}\alpha(\ln\,\mu(\sigma,F))&\le(1+o(1))\alpha(\sigma\alpha^{-1}(\varrho\beta(\sigma)))=\\&=(1+o(1))\alpha(\exp\{\ln\,\sigma+\ln\,\alpha^{-1}(\varrho\beta(\sigma))\})\le\\&\le(1+o(1))\alpha(\exp\{2\max\{\ln\,\sigma,\,\ln\,\alpha^{-1}(\varrho\beta(\sigma))\}\})=\\&=(1+o(1))\alpha(\exp\{\max\{\ln\,\sigma,\,\ln\,\alpha^{-1}(\varrho\beta(\sigma)\}\})=\\&=(1+o(1))\max\{\alpha(\sigma),\,\varrho\beta(\sigma)\})\le\\&\le(1+o(1))(\alpha(\sigma)+\varrho\beta(\sigma)\})=\\&=(1+o(1))\varrho\beta(\sigma)),\qquad\sigma\to+\infty,\end{aligned}$$

whence $\varrho_{\alpha,\beta}[\ln\,\mu,F]\le\varrho$, which is impossible. Thus, $\varrho_{\alpha,\beta}[\ln\,\mu,F]=\varrho_{\alpha,\beta}[\Lambda,F]$. The proof of the equality $\lambda_{\alpha,\beta}[\ln\,\mu,F]=\lambda_{\alpha,\beta}[\Lambda,F]$ is similar. □

Lemma 3. *If $\alpha\in L^0$ and $\beta\in L^0$ then $\varrho_{\alpha,\beta}[F]=\varrho_{\alpha,\beta}[F']$ and $\lambda_{\alpha,\beta}[F]=\lambda_{\alpha,\beta}[F']$.*

Proof. Since [7] for $\sigma<+\infty$ and $0<\delta(\sigma)<+\infty$

$$M(\sigma,F')\le\frac{M(\sigma+\delta(\sigma),F)}{\delta(\sigma)}\tag{9}$$

and for $\sigma_0<\sigma$

$$M(\sigma,F)\le(\sigma-\sigma_0)M(\sigma,F')+M(\sigma_0,F),\tag{10}$$

using $\delta(\sigma)=1$ and $\sigma_0=0$ we have

$$(1+o(1))\ln\,M(\sigma,F)\le\ln\,M(\sigma,F')\le\ln\,M(\sigma+1,F),\quad\sigma\to+\infty,$$

because for every entire Dirichlet series $\ln\,\sigma=o(\ln\,M(\sigma,F))$ as $\sigma\to+\infty$. Since $\alpha\in L^0$ and $\beta\in L^0$, we get $\varrho_{\alpha,\beta}[\ln\,\mu]=\varrho_{\alpha,\beta}[\Lambda]$ and $\lambda_{\alpha,\beta}[\ln\,\mu]=\lambda_{\alpha,\beta}[\Lambda]$. □

Using Lemma 1 we prove the following statement.

Proposition 3. *Let the functions α, β and the sequence (λ_k) satisfy the conditions of Lemma 1, $A[F]=+\infty$ and $-\infty<A[G]\le\overline{A}[G]<+\infty$. Then $\varrho_{\alpha,\beta}[F*G]=\varrho_{\alpha,\beta}[F]$ and if, moreover, $\alpha(\lambda_{k+1})\sim\alpha(\lambda_k)$, $\kappa_k[F]\nearrow+\infty$ and $\kappa_k[G]\nearrow A[G]$ as $k_0\le k\to\infty$ then $\lambda_{\alpha,\beta}[F*G]=\lambda_{\alpha,\beta}[F]$*

Proof. Clearly, if $A[F]=+\infty$ then $\dfrac{1}{\lambda_k}\ln\,\dfrac{1}{|f_k|}\to+\infty$ as $k\to\infty$. On the other hand, since $-\infty<A[G]\le\overline{A}[G]<+\infty$, we have $\dfrac{1}{\lambda_k}\ln\,\dfrac{1}{|g_k|}=O(1)$ as $k\to\infty$. Therefore,

$$\begin{aligned}\beta\left(\frac{1}{\lambda_k}\ln\,\frac{1}{|f_k|}+\frac{1}{\lambda_k}\ln\,\frac{1}{|g_k|}\right)&=\beta\left(\frac{1}{\lambda_k}\ln\,\frac{1}{|f_k|}+O(1)\right)=\\&=\beta\left(\frac{1+o(1)}{\lambda_k}\ln\,\frac{1}{|f_k|}\right)=\\&=(1+o(1))\beta\left(\frac{1}{\lambda_k}\ln\,\frac{1}{|f_k|}\right),\qquad k\to\infty,\end{aligned}$$

and by Lemma 1

$$\varrho_{\alpha,\beta}[F*G]=\varlimsup_{k\to\infty}\frac{\alpha(\lambda_k)}{\beta\left(\dfrac{1}{\lambda_k}\ln\dfrac{1}{|f_kg_k|}\right)}=\varlimsup_{k\to\infty}\frac{\alpha(\lambda_k)}{\beta\left(\dfrac{1}{\lambda_k}\ln\dfrac{1}{|f_k|}\right)}=\varrho_{\alpha,\beta}[F*G]$$

and similarly $\lambda_{\alpha,\beta}[F*G]=\lambda_{\alpha,\beta}[F]$. □

Lemma 3 implies the following statement.

Proposition 4. *If $\alpha\in L^0$ and $\beta\in L^0$ then*

$$\varrho_{\alpha,\beta}[F*G]=\varrho_{\alpha,\beta}[(F*G)^{(n)}]=\varrho_{\alpha,\beta}[F^{(n)}*G^{(n)}]$$

and

$$\lambda_{\alpha,\beta}[F*G]=\lambda_{\alpha,\beta}[(F*G)^{(n)}]=\lambda_{\alpha,\beta}[F^{(n)}*G^{(n)}]$$

for each $n\geq 1$.

Indeed, by Lemma 3 we have that

$$\varrho_{\alpha,\beta}[F*G]=\varrho_{\alpha,\beta}[(F*G)'] \quad\text{and}\quad \lambda_{\alpha,\beta}[F*G]=\lambda_{\alpha,\beta}[(F*G)'],$$

that is

$$\varrho_{\alpha,\beta}[F*G]=\varrho_{\alpha,\beta}[(F*G)^{(n)}] \quad\text{and}\quad \lambda_{\alpha,\beta}[F*G]=\lambda_{\alpha,\beta}[(F*G)^{(n)}]$$

for each $n\geq 1$, and since $F^{(n)}*G^{(n)}=(F*G)^{(2n)}$, we have that

$$\varrho_{\alpha,\beta}[F*G]=\varrho_{\alpha,\beta}[F^{(n)}*G^{(n)}] \quad\text{and}\quad \lambda_{\alpha,\beta}[F*G]=\lambda_{\alpha,\beta}[F^{(n)}*G^{(n)}].$$

3. Behavior of the maximal terms of Hadamard compositions

The following is the main result in the section.

Theorem 1. *Let $\alpha(e^x)\in L_{si}$, $\beta\in L^0$, $\dfrac{d\beta^{-1}(c\alpha(x))}{d\ln x}=O(1)$ as $x\to+\infty$ and (4) holds for each $c\in(0,+\infty)$. If $A[F]=+\infty$ and $-\infty<A[G]\leq\overline{A}[G]<+\infty$ then for $n\in\mathbb{Z}_+$, $m\in\mathbb{N}$ and $m>n$*

$$\varlimsup_{\sigma\to+\infty}\frac{1}{\beta(\sigma)}\alpha\left(\frac{\mu(\sigma,(F*G)^{(m)})}{\mu(\sigma,(F*G)^{(n)})}\right)=\varrho_{\alpha\beta}[F]. \tag{11}$$

If, moreover, $\alpha(\lambda_{k+1})\sim\alpha(\lambda_k)$, $\varkappa_k[F]\nearrow+\infty$ and $\varkappa_k[G]\nearrow A[G]$ as $k_0\leq k\to\infty$ then

$$\varliminf_{\sigma\to+\infty}\frac{1}{\beta(\sigma)}\alpha\left(\frac{\mu(\sigma,(F*G)^{(m)})}{\mu(\sigma,(F*G)^{(n)})}\right)=\lambda_{\alpha\beta}[F]. \tag{12}$$

Proof. The following inequalities proved in [7] play an important role in the proof of Theorem 1

$$\Lambda^{m-n}(\sigma,(F*G)^{(n)})\leq\frac{\mu(\sigma,(F*G)^{(m)})}{\mu(\sigma,(F*G)^{(n)})}\leq\Lambda^{m-n}(\sigma,(F*G)^{(m)}) \tag{13}$$

for $\sigma<\sigma_a[F*G]$. Since $\alpha(e^x)\in L_{si}$, we have

$$\begin{aligned}\alpha(\Lambda^{m-n}(\sigma,(F*G)^{(n)}))&=\alpha(\exp\{(m-n)\ln\Lambda(\sigma,(F*G)^{(n)})\})=\\&=(1+o(1))\alpha(\exp\{\ln\Lambda(\sigma,(F*G)^{(n)})\})=\\&=(1+o(1))\alpha(\Lambda(\sigma,(F*G)^{(n)})),\qquad\sigma\to+\infty,\end{aligned}$$

and, therefore, (13) implies

$$\alpha(\Lambda(\sigma,(F*G)^{(n)}))\le(1+o(1))\alpha\left(\frac{\mu(\sigma,(F*G)^{(m)})}{\mu(\sigma,(F*G)^{(n)})}\right)\le\alpha(\Lambda(\sigma,(F*G)^{(m)}))$$

as $\sigma\to+\infty$, whence

$$\varrho_{\alpha\beta}[\Lambda,(F*G)^{(n)}]\le\varlimsup_{\sigma\to+\infty}\frac{1}{\beta(\sigma)}\alpha\left(\frac{\mu(\sigma,(F*G)^{(m)})}{\mu(\sigma,(F*G)^{(n)})}\right)\le \le\varrho_{\alpha\beta}[\Lambda,(F*G)^{(m)}] \tag{14}$$

and

$$\lambda_{\alpha\beta}[\Lambda,(F*G)^{(n)}]\le\varliminf_{\sigma\to+\infty}\frac{1}{\beta(\sigma)}\alpha\left(\frac{\mu(\sigma,(F*G)^{(m)})}{\mu(\sigma,(F*G)^{(n)})}\right)\le \le\lambda_{\alpha\beta}[\Lambda,(F*G)^{(m)}]. \tag{15}$$

We remark that the condition $\dfrac{d\beta^{-1}(c\alpha(x))}{d\ln x}=O(1)$ as $x\to+\infty$ for each $c\in(0,+\infty)$ implies the condition $\alpha(x)=o(\beta(x))$ as $x\to+\infty$. Therefore, applying Lemma 2, Lemma 1, Proposition 4 and Proposition 3 consequently, we obtain $\varrho_{\alpha\beta}[\Lambda,(F*G)^{(n)}]=$ $=\varrho_{\alpha\beta}[\ln\mu,(F*G)^{(n)}]=\varrho_{\alpha\beta}[(F*G)^{(n)}]=\varrho_{\alpha\beta}[F*G]=\varrho_{\alpha\beta}[F]$ and similarly $\lambda_{\alpha\beta}[\Lambda,(F*G)^{(n)}]=\lambda_{\alpha\beta}[F]$. Therefore, from (14) and (15) we get (11) and (12). □

Choosing $m=2n$ we obtain the following corollary.

Corollary 1. *Let the functions α, β and the sequence (λ_k) satisfy the conditions of Theorem 1, $A[F]=+\infty$ and $-\infty<A[G]\le\overline{A}[G]<+\infty$ then for $n\in\mathbb{N}$*

$$\varlimsup_{\sigma\to+\infty}\frac{1}{\beta(\sigma)}\alpha\left(\frac{\mu(\sigma,F^{(n)}*G^{(n)})}{\mu(\sigma,(F*G)^{(n)})}\right)=\varrho_{\alpha\beta}[F].$$

If, moreover, $\alpha(\lambda_{k+1})\sim\alpha(\lambda_k)$, $\kappa_k[F]\nearrow+\infty$ and $\kappa_k[G]\nearrow A[G]$ as $k_0\le k\to\infty$ then

$$\varliminf_{\sigma\to+\infty}\frac{1}{\beta(\sigma)}\alpha\left(\frac{\mu(\sigma,F^{(n)}*G^{(n)})}{\mu(\sigma,(F*G)^{(n)})}\right)=\lambda_{\alpha\beta}[F].$$

4. Hadamard compositions of the finite R-order

If we choose $\alpha(x)=\ln x$ and $\beta(x)=x$ for $x\ge3$ then from (3) we obtain the definition of the R-order

$$\varrho_R[F]:=\varlimsup_{\sigma\to+\infty}\frac{\ln\ln M(\sigma,F)}{\sigma}$$

and the lower R-order

$$\lambda_R[F]:=\varliminf_{\sigma\to+\infty}\frac{\ln\ln M(\sigma,F)}{\sigma}$$

introduced by J. Ritt [12] for a function $F\in S(\Lambda,+\infty)$.

The functions $\alpha(x)=\ln x$ and $\beta(x)=x$ satisfy the conditions of Lemmas 1 and 3 and do not satisfy the condition $\alpha(e^x)\in L_{si}$ of Lemma 2. But it follows from (8) that $\varrho_R[\Lambda,F]=\varrho_R[\ln\mu,F]$ and $\lambda_R[\Lambda,F]=\lambda_R[\ln\mu,F]$. Therefore, as in the proof of

Theorem 1, we have $\varrho_R[\Lambda,(F*G)^{(n)}]=\varrho_R[F]$ and $\lambda_R[\Lambda,(F*G)^{(n)}]=\lambda_R[F]$. On the other hand, from (13) we get

$$(m-n)\ln\Lambda(\sigma,(F*G)^{(n)})\le\ln\frac{\mu(\sigma,(F*G)^{(m)})}{\mu(\sigma,(F*G)^{(n)})}\le \le(m-n)\ln\Lambda^{m-n}(\sigma,(F*G)^{(m)}) \tag{16}$$

and, thus, the following theorem is true.

Theorem 2. *If $A[F]=+\infty$, $-\infty<A[G]\le\overline{A}[G]<+\infty$), and $\ln k=o(\lambda_k\ln\lambda_k)$ as $k\to\infty$. Then for $n\in\mathbb{Z}_+$, $m\in\mathbb{N}$ and $m>n$*

$$\varlimsup_{\sigma\to+\infty}\frac{1}{\sigma}\ln\frac{\mu(\sigma,(F*G)^{(m)})}{\mu(\sigma,(F*G)^{(n)})}=(m-n)\varrho_R[F].$$

If, moreover, $\ln\lambda_{k+1}\sim\ln\lambda_k$, $\kappa_k[F]\nearrow+\infty$ and $\kappa_k[G]\nearrow A[G]$ as $k_0\le k\to\infty$ then

$$\varliminf_{\sigma\to+\infty}\frac{1}{\sigma}\ln\frac{\mu(\sigma,(F*G)^{(m)})}{\mu(\sigma,(F*G)^{(n)})}=(m-n)\lambda_R[F].$$

If we choose $m=2n+2$ then from Theorem 2 we obtain the following analogue of the above-mentioned result of M.K. Sen.

Corollary 2. *$A[F]=+\infty$, $-\infty<A[G]\le\overline{A}[G]<+\infty$, and $\ln k=o(\lambda_k\ln\lambda_k)$ as $k\to\infty$. Then for $n\in\mathbb{Z}_+$*

$$\varlimsup_{\sigma\to+\infty}\frac{1}{\sigma}\ln\frac{\mu(\sigma,F^{(n+1)}*G^{(n+1)})}{\mu(\sigma,(F*G)^{(n)})}=(n+2)\varrho_R[F].$$

If, moreover, $\ln\lambda_{k+1}\sim\ln\lambda_k$, $\kappa_k[F]\nearrow+\infty$, and $\kappa_k[G]\nearrow A[G]$ as $k_0\le k\to\infty$ then

$$\varliminf_{\sigma\to+\infty}\frac{1}{\sigma}\ln\frac{\mu(\sigma,F^{(n+1)}*G^{(n+1)})}{\mu(\sigma,(F*G)^{(n)})}=(n+2)\lambda_R[F].$$

Let now $0<\varrho_R[F]<+\infty$. If we choose $\alpha(x)=x$ and $\beta(x)=\exp\{\varrho_R[F]x\}$ for $x\ge0$ then from (3) we obtain the definition of the R-type,

$$T_R[F]=\varlimsup_{\sigma\to+\infty}\frac{\ln M(\sigma,F)}{\exp\{\varrho_R[F]\sigma\}},$$

and the lover R-type,

$$t_R[F]=\varliminf_{\sigma\to+\infty}\frac{\ln M(\sigma,F)}{\exp\{\varrho_R[F]\sigma\}}.$$

It is clear that the functions $\alpha(x)=x$ and $\beta(x)=\exp\{\varrho_R[F]x\}$ do not satisfy the conditions of Lemma 1, but the following lemma is true (see for example [10], [12], [13]).

Lemma 4. *If $F\in(\Lambda,+\infty)$ and $\ln k=o(\lambda_k)$ as $k\to\infty$ then*

$$T_R[F]=T_R[\ln\mu,F]=\varlimsup_{k\to\infty}\frac{\lambda_k}{e\varrho_R[F]}|f_k|^{\varrho_R[F]/\lambda_k}.$$

If, moreover, $\lambda_{k+1}\sim\lambda_k$ and $\kappa_k[F]\nearrow+\infty$ as $k_0\le k\to\infty$ then

$$t_R[F]=t_R[\ln\mu,F]=\varliminf_{k\to\infty}\frac{\lambda_k}{e\varrho_R[F]}|f_k|^{\varrho_R[F]/\lambda_k}.$$

The following lemma indicates the connection between the growth of $\ln\,\mu(\sigma, F)$ and $\Lambda(\sigma, F)$ in terms of R-types.

Lemma 5. *Let $F \in (\Lambda, +\infty)$ and $\ln\, k = o(\lambda_k)$ as $k \to \infty$. Then*

$$\frac{T_R[\Lambda, F]}{e\varrho_R[F]} \le T_R[\ln\,\mu, F] \le \frac{T_R[\Lambda, F]}{\varrho_R[F]} \tag{17}$$

and

$$\frac{t_R[\Lambda, F]}{e\varrho_R[F]} \le t_R[\ln\,\mu, F] \le \frac{T_R[\Lambda, F]}{\varrho_R[F]} \ln \frac{e\varrho_R[F] T_R[\ln\,\mu, F]}{T_R[\Lambda, F]}. \tag{18}$$

Proof. From (7) for $\sigma \ge 1/\varrho_R[F]$ we have

$$\ln\,\mu(\sigma, F) - \ln\,\mu(0, F) \ge \int\limits_{\sigma-1/\varrho_R[F]}^{\sigma} \Lambda(x)dx \ge \frac{\Lambda(\sigma - 1/\varrho_R[F])}{\varrho_R[F]},$$

i.e.,

$$T_R[\ln\,\mu, F] \ge \varlimsup_{\sigma\to+\infty} \frac{\Lambda(\sigma - 1/\varrho_R[F])}{\varrho_R[F]\exp\{\varrho_R[F]\sigma\}} = \varlimsup_{\sigma\to+\infty} \frac{\Lambda(\sigma - 1/\varrho_R[F])}{e\varrho_R[F]\exp\{\varrho_R[F](\sigma - 1/\varrho_R[F])\}},$$

whence $T_R[\ln\,\mu, F] \ge \dfrac{T_R[\Lambda, F]}{e\varrho_R[F]}$. Similarly, $t_R[\ln\,\mu, F] \ge \dfrac{t_R[\Lambda, F]}{e\varrho_R[F]}$. Thus, the inequalities on the left side in (17) and (18) are proved.

On the other hand, if $T_R[\Lambda, F] < +\infty$ then $\Lambda(\sigma) \le T\exp\{\varrho_R[F]\sigma\}$ for every $T > T_R[\Lambda.F]$ and all $\sigma \ge \sigma_0(T)$. Therefore,

$$\ln\,\mu(\sigma, F) - \ln\,\mu(\sigma_0(T), F) \le T \int\limits_{\sigma_0(T)}^{\sigma} \exp\{\varrho_R[F]x\}dx =$$
$$= \frac{T}{\varrho_R[F]}(\exp\{\varrho_R[F]\sigma\} - \exp\{\varrho_R[F]\sigma_0(T)\}),$$

whence $T_R[\ln\,\mu, F] \le T/\varrho_R[F]$, i.e. in view of the arbitrariness of T we get $T_R[\ln\,\mu, F] \le T_R[\Lambda, F]/\varrho_R[F]$.

Finally, suppose that $t_R[\ln\mu, F] > 0$ and $T_R[\Lambda, F] > 0$. Then for every $t \in (0, t_R[\ln\,\mu, F])$ and $T \in (0, T_R[\Lambda, F])$ there exists an unbounded set $E \subset [0, +\infty)$ such that $\ln\,\mu(\sigma, F) \ge t\exp\{\varrho[F]\sigma\}$ and $\Lambda(\sigma) \ge T\exp\{\varrho_R[F]\sigma\}$. Therefore, for $\sigma^* \in E$ and $\sigma > \sigma^*$

$$\ln\,\mu(\sigma, F) = \ln\,\mu(\sigma^*, F) + \int\limits_{\sigma^*}^{\sigma} \Lambda(x, F)dx$$
$$\ge \ln\,\mu(\sigma^*, F) + \Lambda(\sigma^*, F) \int\limits_{\sigma^*}^{\sigma} dx \ge$$
$$\ge t\exp\{\varrho_R[F]\sigma^*\} + (\sigma - \sigma^*)T\exp\{\varrho_R[F]\sigma^*\}.$$

Therefore,

$$\frac{\ln \mu(\sigma, F)}{\exp\{\varrho_R[F]\sigma\}} \geq \frac{t + (\sigma - \sigma^*)T}{\exp\{\varrho_R[F](\sigma - \sigma^*)\}}.$$

Since the maximum of the function $\varphi(x) = \dfrac{t + Tx}{\exp\{\varrho_R[F]x\}}$ is reached at the point $x = \dfrac{T - t\varrho_R[F]}{T\varrho_R[F]}$, we obtain $T_R[\ln \mu] \geq \dfrac{T}{e\varrho_R[F]} \exp\left\{\dfrac{\varrho_R[F]t}{T}\right\}$ and in view of the arbitrariness of t and T we get

$$T_R[\ln \mu, F] \geq \frac{T_R[\Lambda, F]}{e\varrho_R[F]} \exp\left\{\frac{\varrho_R[F]t_R[\ln \mu, F]}{T_R[\Lambda, F]}\right\},$$

whence the right side of (18) follows. The proof of Lemma 5 is complete. □

Lemma 6. *For every entire Dirichlet series* (1) $T_R[F] = T_R[F']$ *and* $t_R[F] = t_R[F']$.

Proof. Choosing $\delta(\sigma) = 1/(\sigma + 1)$ for $\sigma \geq 0$ from (9) we obtain

$$\frac{\ln M(\sigma, F')}{\exp\{\sigma\varrho_R[F]\}} \leq \frac{\ln M(\sigma + 1/(\sigma + 1), F') + \ln(\sigma + 1)}{\exp\{\sigma\varrho_R[F]\}} =$$
$$= \frac{\ln M(\sigma + 1/(\sigma + 1), F')}{\exp\{(\sigma + 1/(\sigma + 1))\varrho_R[F]\}} \exp\left\{\frac{\varrho_R[F]}{\sigma + 1}\right\} + \frac{\ln(\sigma + 1)}{\exp\{\sigma\varrho_R[F]\}},$$

whence $T_R[F'] \leq T_R[F]$ and $t_R[F'] \leq t_R[F]$. On the other hand, in view of (10) $\ln M(\sigma, F) \leq (1 + o(1)) \ln M(\sigma, F)$ as $\sigma \to +\infty$, whence $T_R[F] \leq T_R[F']$ and $t_R[F] \leq t_R[F']$. □

Using Lemma 4 we prove the following statement.

Proposition 5. *Let* $A[F] = +\infty$, $-\infty < A[G] \leq \overline{A}[G] < +\infty$ *and* $\ln k = o(\lambda_k)$ *as* $k \to \infty$. *Then*

$$T_R[F] \exp\{-\overline{A}[G]\varrho_R[F]\} \leq T_R[F * G] \leq T_R[F] \exp\{-A[G]\varrho_R[F]\} \tag{19}$$

and if, moreover, $\lambda_{k+1} \sim \lambda_k$, $\kappa_k[F] \nearrow +\infty$ *and* $\kappa_k[G] \nearrow A[G]$ *as* $k_0 \leq k \to \infty$ *then*

$$t_R[F] \exp\{-\overline{A}[G]\varrho_R[F]\} \leq t_R[F * G] \leq t_R[F] \exp\{-A[G]\varrho_R[F]\}. \tag{20}$$

Proof. By Proposition 3 $\varrho_R[F * G] = \varrho_R[F]$. Therefore, by Lemma 4

$$T_R[F * G] = \overline{\lim_{k\to\infty}} \frac{\lambda_k}{e\varrho_R[F * G}]|f_k g_k|^{\varrho_R[F*G]/\lambda_k} =$$
$$= \overline{\lim_{k\to\infty}} \frac{\lambda_k}{e\varrho_R[F]}|f_k|^{\varrho_R[F]/\lambda_k} \exp\left\{-\varrho_R[F]\frac{1}{\lambda_k} \ln \frac{1}{|g_k|}\right\},$$

whence

$$T_R[F * G] \leq \overline{\lim_{k\to\infty}} \frac{\lambda_k}{e\varrho_R[F]}|f_k|^{\varrho_R[F]/\lambda_k} \overline{\lim_{k\to\infty}} \exp\left\{-\varrho_R[F]\frac{1}{\lambda_k} \ln \frac{1}{|g_k|}\right\} =$$
$$= T_R[F] \exp\left\{-\varrho_R[F] \underline{\lim_{k\to\infty}} \frac{1}{\lambda_k} \ln \frac{1}{|g_k|}\right\} =$$
$$= T_R[F] \exp\{-A[G]\varrho_R[F]\}.$$

and

$$\begin{aligned}T_R[F*G] &\geq \varlimsup_{k\to\infty}\frac{\lambda_k}{e\varrho_R[F]}|f_k|^{\varrho_R[F]/\lambda_k}\varliminf_{k\to\infty}\exp\left\{-\varrho_R[F]\frac{1}{\lambda_k}\ln\frac{1}{|g_k|}\right\}=\\ &=T_R[F]\exp\left\{-\varrho_R[F]\varlimsup_{k\to\infty}\frac{1}{\lambda_k}\ln\frac{1}{|g_k|}\right\}=\\ &=T_R[F]\exp\{-\overline{A}[G]\varrho_R[F]\},\end{aligned}$$

i.e. we get (19). The proof of (20) is similar. □

Finally, Lemma 6 implies the following statement.

Proposition 6. *The equalities*

$$T_R[F*G]=T_R[(F*G)^{(n)}]=T_R[F^{(n)}*G^{(n)}]$$

and

$$t_R[F*G]=t_R[(F*G)^{(n)}]=t_R[F^{(n)}*G^{(n)}]$$

are true for each $n\geq 1$.

Therefore, the following theorem is true.

Theorem 3. *Let* $A[F]=+\infty$, $-\infty<A[G]\leq\overline{A}[G]<+\infty$ *and* $\ln k=o(\lambda_k)$ *as* $k\to\infty$. *Then for* $n\in\mathbb{Z}_+$, $m\in\mathbb{N}$ *and* $m>n$

$$\tag{21}\begin{aligned}\frac{\varrho_R[F]T_R[F*G]}{\exp\{\overline{A}[G]\varrho_R[F]\}}&\leq\varlimsup_{\sigma\to+\infty}\frac{1}{\exp\{\varrho_R[F]\sigma\}}\sqrt[m-n]{\frac{\mu(\sigma,(F*G)^{(m)})}{\mu(\sigma,(F*G)^{(n)})}}\leq\\ &\leq\frac{e\varrho_R[F]T_R[F*G]}{\exp\{A[G]\varrho_R[F]\}}.\end{aligned}$$

Proof. From (13) it follows that

$$\tag{22}\begin{aligned}T_R[\Lambda,(F*G)^{(n)}]&\leq\varlimsup_{\sigma\to+\infty}\frac{1}{\exp\{\varrho_R[F]\sigma\}}\sqrt[m-n]{\frac{\mu(\sigma,(F*G)^{(m)})}{\mu(\sigma,(F*G)^{(n)})}})\leq\\ &\leq T_R[\Lambda,(F*G)^{(m)}].\end{aligned}$$

Using Proposition 6, Lemmas 5 and 4 from (22) we get (21). □

Remark 1. Similarly, we can prove that if the conditions of Theorem 3 are satisfied and, moreover, $\lambda_{k+1}\sim\lambda_k$, $\kappa_k[F]\nearrow+\infty$ and $\kappa_k[G]\nearrow A[G]$ as $k_0\leq k\to\infty$ then

$$\varliminf_{\sigma\to+\infty}\frac{1}{\exp\{\varrho_R[F]\sigma\}}\sqrt[m-n]{\frac{\mu(\sigma,(F*G)^{(m)})}{\mu(\sigma,(F*G)^{(n)})}}\leq\frac{e\varrho_R[F]t_R[F*G]}{\exp\{A[G]\varrho_R[F]\}}$$

We were not able to obtain a lower estimate for this $\varliminf$, because there is no such an estimate for $t_R(\Lambda)$.

5. Hadamard compositions of the finite logarithmic order

In the theory of entire Dirichlet series, the logarithmic order

$$\varrho_l[F] :== \varlimsup_{\sigma\to+\infty} \frac{\ln\ln M(\sigma,F)}{\ln\sigma}$$

and lower order

$$\lambda_l[F] := \varliminf_{\sigma\to+\infty} \frac{\ln\ln M(\sigma,F)}{\ln\sigma}$$

are also used. We remark that $\lambda_l[F]\ge 1$ for each entire Dirichlet series.

The function $\alpha(x)=\beta(x)=\ln x$ not hold the condition of Lemma 1, but the following statement is true [13].

Lemma 7. *If* $F\in(\Lambda,+\infty)$ *and*

$$\varlimsup_{k\to\infty}\frac{\ln\ln k}{\ln\lambda_k}<1 \tag{23}$$

then $\varrho_l[F]=\varlimsup\limits_{k\to\infty}\dfrac{\ln\lambda_k}{\ln\left(\frac{1}{\lambda_k}\ln\frac{1}{|f_k|}\right)}+1$. *If, moreover,* $\ln\lambda_{k+1}\sim\ln\lambda_k$ *and* $\varkappa_k[F]\nearrow+\infty$ *as* $k_0\le k\to\infty$ *then* $\lambda_l[F]=\varliminf\limits_{k\to\infty}\dfrac{\ln\lambda_k}{\ln\left(\frac{1}{\lambda_k}\ln\frac{1}{|f_n|}\right)}+1$.

As in the proof of Proposition 3 using Lemma 7 we get the following statement.

Proposition 7. *Let* $A[F]=+\infty$, $-\infty<A[G]\le\overline{A}[G]<+\infty$ *and* (23) *holds. Then* $\varrho_l[F*G]=\varrho_l[F]$. *If, moreover,* $\ln\lambda_{k+1}\sim\ln\lambda_k$, $\varkappa_k[F]\nearrow+\infty$ *and* $\varkappa_k[G]\nearrow A[G]$ *as* $k_0\le k\to\infty$ *then* $\lambda_l[F*G]=\lambda_l[F]$.

From (9) with $\delta(\sigma)=1$ and (10) we obtain $\varrho_l[F']=\varrho_l[F]$ and $\lambda_l[F']=\lambda_l[F]$. From (8) with $\varepsilon=1$ we obtain

$$\frac{\sigma}{2}\Lambda\left(\frac{\sigma}{2},F\right)\le\ln\mu(\sigma,F)-\ln\mu(0,F)\le\sigma\Lambda(\sigma),$$

whence $\varrho_l[\ln\mu,F]-1=\varrho_l[\Lambda,F]$ and $\lambda_l[\ln\mu,F]-1=\lambda_l[\Lambda,F]$. Finally, (16) implies the inequalities

$$(m-n)\varrho_l(\Lambda,(F*G)^{(n)})\le\varlimsup_{\sigma\to+\infty}\frac{1}{\ln\sigma}\ln\frac{\mu(\sigma,(F*G)^{(m)})}{\mu(\sigma,(F*G)^{(n)})}\le$$
$$\le(m-n)\varrho_l(\Lambda,(F*G)^{(m)})$$

and

$$(m-n)\lambda_l(\Lambda,(F*G)^{(n)})\le\varliminf_{\sigma\to+\infty}\frac{1}{\ln\sigma}\ln\frac{\mu(\sigma,(F*G)^{(m)})}{\mu(\sigma,(F*G)^{(n)})}\le$$
$$\le(m-n)\lambda_l(\Lambda,(F*G)^{(m)}).$$

Therefore, as usual, we arrive at the following theorem.

Theorem 4. *Let* $A[F] = +\infty$, $-\infty < A[G] \le \overline{A}[G] < +\infty$ *and* (23) *holds. Then for* $n \in \mathbb{Z}_+$, $m \in \mathbb{N}$ *and* $m > n$

$$\varlimsup_{\sigma\to+\infty} \frac{1}{\ln\sigma} \ln \frac{\mu(\sigma,(F*G)^{(m)})}{\mu(\sigma,(F*G)^{(n)})} = (m-n)(\varrho_l[F]-1).$$

If, moreover, $\ln\lambda_{k+1} \sim \ln\lambda_k$, $\varkappa_k[F] \nearrow +\infty$ *and* $\varkappa_k[G] \nearrow A[G]$ *as* $k_0 \le k \to \infty$ *then*

$$\varliminf_{\sigma\to+\infty} \frac{1}{\ln\sigma} \ln \frac{\mu(\sigma,(F*G)^{(m)})}{\mu(\sigma,(F*G)^{(n)})} = (m-n)(\lambda_l[F]-1).$$

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ПРО АДАМАРОВІ КОМПОЗИЦІЇ ЦІЛОГО РЯДУ ДІРІХЛЕ ТА РЯДУ ДІРІХЛЕ, АБСОЛЮТНО ЗБІЖНОГО У ПІВПЛОЩИНІ

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Для степеневих рядів $f(z)=\sum\limits_{k=0}^{\infty} f_k z^k$ і $g(z)=\sum\limits_{k=0}^{\infty} g_k z^k$ із радіусами збіжності $R[f]$ і $R[g]$ ряд $(f*g)(z)=\sum\limits_{k=0}^{\infty} f_k g_k z^k$ називається *адамаровою композицією*. Для $0\le r<R[f]$ нехай $\mu_f(r)=\max\{|f_k|r^k:\,k\ge 0\}$ – максимальний член степеневого розвинення функції f. Вивчаючи зв'язок між зростанням максимальних членів похідних адамарової композиції двох цілих функцій f та g і адамаровою композицією їх похідних М. Сен зокрема довів, що якщо функція $(f*g)$ має порядок ϱ і нижній порядок λ, то для кожного $\varepsilon>0$ і всіх $r\ge r_0(\varepsilon)$

$$r^{(n+2)\lambda-1-\varepsilon}\le\frac{\mu_{f^{(n+1)}*g^{(n+1)}}(r)}{\mu_{(f*g)^{(n)}}(r)}\le r^{(n+2)\varrho-1+\varepsilon}.$$

Оскільки ряди Діріхле з додатними зростаючими до $+\infty$ показниками є прямим узагальненням степеневих рядів, то природно постає питання про подібні результати для адамарової композиції таких рядів. Отже, нехай $\Lambda=(\lambda_k)$ – зростаюча до $+\infty$ послідовність невід'ємних чисел ($\lambda_0=0$), і $S(\Lambda,A)$ – клас рядів Діріхле $F(s)=\sum\limits_{k=0}^{\infty} f_k\exp\{s\lambda_k\},(s=\sigma+it)$, з показниками Λ і абсцисою абсолютної збіжності $\sigma_a[F]=A$. Якщо $F\in(\Lambda,A_1)$ і $G(s)=\sum\limits_{k=0}^{\infty} g_k\exp\{s\lambda_k\}\in(\Lambda,A_2)$, то ряд Діріхле

$$(F*G)(s)=\sum_{k=0}^{\infty} f_k g_k\exp\{s\lambda_k\}$$

називається *адамаровою композицією* функцій F та G.
Для ряду Діріхле $F(s)$ з $\sigma_a[F]=A[F]=A>-\infty$ для $\sigma<A$ максимальним членом називатимемо $\mu(\sigma,F)=\max\{|f_k|\exp\{\sigma\lambda_k\}:\,k\ge 0\}$. Відомо, що для $n\in\mathbb{Z}_+$, $m\in\mathbb{N}$ і $m>n$, якщо $\sigma_a[F]=\sigma_a[G]=+\infty$ і $\ln k=o(\lambda_k\ln\lambda_k)$ при $k\to\infty$, то

$$\varlimsup_{\sigma\to+\infty}\frac{1}{\sigma}\ln\frac{\mu(\sigma,(F*G)^{(m)})}{\mu(\sigma,(F*G)^{(n)})}=(m-n)\varrho_R[f*G]$$

і (якщо $\varrho_R[f*G]<+\infty$)

$$\varliminf_{\sigma\to+\infty}\frac{1}{\sigma}\ln\frac{\mu(\sigma,(F*G)^{(m)})}{\mu(\sigma,(F*G)^{(n)})}=(m-n)\lambda_R[f*G],$$

де $\varrho_R[f]$ і $\lambda_R[f]$ відповідно R-порядок та нижній R-порядок цілого ряду Діріхле. Якщо $\sigma_a[F] = \sigma_a[G] = 0$ і $\ln k = o(\lambda_k / \ln \lambda_k)$ при $k \to \infty$, то

$$\overline{\lim_{\sigma\uparrow 0}} |\sigma| \ln \frac{\mu(\sigma, (F*G)^{(m)})}{\mu(\sigma, (F*G)^{(n)})} = (m-n)\varrho^{(0)}[f*G]$$

і

$$\underline{\lim_{\sigma\uparrow 0}} |\sigma| \ln \frac{\mu(\sigma, (F*G)^{(m)})}{\mu(\sigma, (F*G)^{(n)})} = (m-n)\lambda^{(0)}[f*G],$$

де $\varrho^{(0)}[f]$ і $\lambda^{(0)}[f]$ відповідно порядок та нижній порядок ряду Діріхле з $\sigma_a[F] = 0$.
У праці отримано аналогічні результати для випадку $\sigma_a[F] = +\infty$ і $\sigma_a[G] \in (-\infty + \infty)$.

Ключові слова: ряд Діріхле, композиція Адамара, узагальнений порядок, максимальний член.