A new simple method for constructing solutions of multidimensional nonlinear d’Alembert equations is proposed.

Let us consider a nonlinear Poincaré-invariant d’Alembert equation

\[ \square u + F(u) = 0, \quad (1) \]

where

\[ \square u = \frac{\partial^2 u}{\partial x_1^2} - \frac{\partial^2 u}{\partial x_2^2} - \cdots - \frac{\partial^2 u}{\partial x_n^2}. \]

\( F(u) \) is an arbitrary smooth function. Papers [1, 5, 6, 7] are devoted to the construction of exact solutions to equation (1) for different restrictions on the function \( F(x) \). Majority of these solutions is invariant with respect to a subgroup of the invariance group of equation (1), i.e., they are Lie solutions. One of the methods for constructing solutions is the method of symmetry reduction of equation (1) to ordinary differential equations. The essence of this method for equation (1) consists in the following.

Equation (1) is invariant under the Poincaré algebra \( AP(1, n) \) with the basis elements

\[ J_{0a} = \partial_0 x_a + x_a \partial_0, \quad J_{ab} = x_b \partial_a - x_a \partial_b, \]

\[ P_0 = \partial_0, \quad P_a = \partial_a \quad (a, b = 1, 2, \ldots, n). \]

Let \( L \) be an arbitrary rank \( n \) subalgebra of the algebra \( AP(1, n) \). The subalgebra \( L \) has two main invariants \( \omega, \varphi = \varphi(x_0, x_1, \ldots, x_n) \). The ansatz \( u = \varphi(\omega) \) corresponding to the subalgebra \( L \) reduces equation (1) to the ordinary differential equation

\[ \frac{\partial^2 \varphi}{\partial \omega^2} + \varphi \frac{\partial \varphi}{\partial \omega} + F(\varphi) = 0, \]

where

\[ (\nabla \omega)^2 \equiv \left( \frac{\partial \omega}{\partial x_0} \right)^2 - \left( \frac{\partial \omega}{\partial x_1} \right)^2 - \cdots - \left( \frac{\partial \omega}{\partial x_n} \right)^2. \]

Such a reduction is called the symmetry reduction, and the ansatz is called the symmetry ansatz. There exist eight types of nonequivalent rank \( n \) subalgebras of the algebra \( AP(1, n) \) [5]. In Table 1, we write out these subalgebras, their invariants and values of \( (\nabla \omega)^2, \square \omega \) for each invariant.
Let us impose the condition on equation (4), under which equation (4) coincides with the reduced equation (3). Under such assumption, equation (4) decomposes into two equations

\[ \phi_{ij} + \frac{1}{w_i^2} \phi_{ii} + F(\psi) = 0, \]

\[ 2\phi_{ij}(\nabla \omega_i, \nabla \omega_j) + \phi_{ij} \omega_i = 0. \]

Equation (5) will be fulfilled for an arbitrary function \( \psi \) if we impose the conditions

\[ \omega_i = 0, \quad \omega_i^2 = 0, \quad (\omega_j) = 0. \]

on the variable \( \omega_j \). Therefore, if we choose the variable \( \omega_j \) such that conditions (6), (7) are satisfied, then the multidimensional equation (1) is reduced to the ordinary differential equation (3) and solutions of the latter equation give us solutions of equation (1). So, the problem of reduction is reduced to the construction of general or partial solutions to system (6), (7).

The overdetermined system (6) is studied in detail in papers [10, 11]. A wide class of solutions to system (6) is constructed in papers [10, 11]. These solutions are constructed in the following way. Let us consider a linear algebraic equation in variables \( x_0, x_1, \ldots, x_n \) with coefficients depending on the unknown \( \omega_j \):

\[ a_0(\omega_j)x_0 - a_1(\omega_j)x_1 - \cdots - a_n(\omega_j)x_n - b(\omega_j) = 0. \]

(8)

Let the coefficients of this equation represent analytic functions of \( \omega_j \) satisfying the condition

\[ [a_0(\omega_j)]^2 - [a_1(\omega_j)]^2 - \cdots - [a_n(\omega_j)]^2 = 0. \]

Suppose that equation (8) is solvable for \( \omega_j \) and let a solution of this equation represent some real or complex function

\[ \omega_j(x_0, x_1, \ldots, x_n). \]

(9)

Then function (9) is a solution to system (6). Single out those solutions (9), that possess the additional property \( \nabla \omega_1 \nabla \omega_2 = 0. \) It is obvious that

\[ \frac{\partial \omega_2}{\partial x_0} = \frac{a_0}{\delta'}, \quad \frac{\partial \omega_2}{\partial x_1} = \frac{a_1}{\delta'}, \quad \ldots, \quad \frac{\partial \omega_2}{\partial x_n} = \frac{a_n}{\delta'}, \]

where

\[ \delta(\omega_j) \equiv a_0(\omega_j)x_0 - a_1(\omega_j)x_1 - \cdots - a_n(\omega_j)x_n - b(\omega_j) \]

and \( \delta' \) is the derivative of \( \delta \) with respect to \( \omega_j \). Since

\[ \frac{\partial \omega_1}{\partial x_0} = \frac{x_0}{\omega_1}, \quad \frac{\partial \omega_1}{\partial x_1} = \frac{x_1}{\omega_1}, \quad \ldots, \quad \frac{\partial \omega_1}{\partial x_n} = \frac{x_n}{\omega_1}. \]
we have

$$\nabla \omega_1 \cdot \nabla \omega_2 = -\frac{1}{\omega_1} \left( a_0 x_0 - a_1 x_1 - \cdots - a_n x_n \right).$$

Hence, with regard for (8), the equality $\nabla \omega_1 \cdot \nabla \omega_2 = 0$ is fulfilled if and only if $b(\omega_2) = 0$.

Therefore, we have constructed the wide class of ansatzes reducing the d'Alembert equation to ordinary differential equations. The arbitrariness in choosing the function $\omega_2$ may be used to satisfy some additional conditions (initial, boundary and so on).

b) The symmetry ansatz $u = \varphi(\omega_1)$, $\omega_1 = (x_1^2 + \cdots + x_l^2)^{1/2}$, $1 \leq l < n-1$, is generalized in the following way. Let $\omega_2$ be an arbitrary solution to the system of equations

$$\frac{\partial^2 \omega}{\partial x_0^2} - \frac{\partial^2 \omega}{\partial x_1^2} + \cdots - \frac{\partial^2 \omega}{\partial x_n^2} = 0,$$

$$\left( \frac{\partial \omega}{\partial x_0} \right)^2 - \left( \frac{\partial \omega}{\partial x_1} \right)^2 + \cdots - \left( \frac{\partial \omega}{\partial x_n} \right)^2 = 0. \quad (10)$$

The ansatz $u = \varphi(\omega_1, \omega_2)$ reduces equation (1) to the equation

$$-\frac{d^2 \varphi}{d \omega_1^2} - k - 1 \frac{d \varphi}{d \omega_1} + F(\varphi) = 0.$$ 

If $l = n-1$, then the ansatz $u = \varphi(\omega_1, \omega_2)$, $\omega_2 = x_0 - x_n$ is a generalization of the symmetry ansatz $u = \varphi(\omega_1)$.

Ansatzes corresponding to subalgebras 2, 6 and 8 in Table 1, are particular cases of the ansatz constructed above. Doing in a similar way, one can obtain wide classes of ansatzes reducing equation (1) to two-dimensional, three-dimensional and so on equations. Let us present some of them.

c) The ansatz $u = \varphi(\omega_1, \ldots, \omega_l, \omega_{l+1})$, where $\omega_1 = x_1, \ldots, \omega_l = x_l$, $\omega_{l+1}$ is an arbitrary solution of system (10), $l \leq n-1$, is a generalization of the symmetry ansatz $u = \varphi(\omega_1, \ldots, \omega_l)$ and reduces equation (1) to the equation

$$\frac{\partial^2 \varphi}{\partial \omega_1^2} - \frac{\partial^2 \varphi}{\partial \omega_2^2} + \cdots - \frac{\partial^2 \varphi}{\partial \omega_n^2} + F(\varphi) = 0.$$

d) The ansatz $u = \varphi(\omega_1, \ldots, \omega_i, \omega_{i+1})$, where $\omega_1 = (x_1^2 + \cdots + x_l^2)^{1/2}$, $\omega_2 = x_{l+1}, \ldots, \omega_i = x_{i+s+1}$, $i \geq 2$, $l + s - 1 \leq n$, $\omega_{i+1}$ is an arbitrary solution of the system

$$\omega_{i+1} = 0, \quad (\nabla \omega_{i+1})^2 = 0, \quad \nabla \omega_i \cdot \nabla \omega_{i+1} = 0, \quad i = 1, 2, \ldots, s. \quad (11)$$

is a generalization of the symmetry ansatz $u = \varphi(\omega_1, \ldots, \omega_i)$ and reduces equation (1) to the equation

$$\varphi_1 - \frac{l}{\omega_1} \varphi_1 - \varphi_2 - \cdots - \varphi_{s+1} + F(\varphi) = 0.$$

Let us construct in the way described above some classes of exact solutions of the equation

$$\square u + \lambda u^k = 0, \quad k \neq 1. \quad (12)$$
The following solution of equation (12) is obtained in paper [7]:

\[ u^{1-k} = \sigma(k, l)(x_1^2 + \cdots + x_l^2), \tag{13} \]

where

\[ \sigma(k, l) = \frac{\lambda(1-k)^2}{2(l-k+2k)}, \quad l = 1, 2, \ldots, n. \]

Solution (13) defines a multiparameter solution set

\[ u^{1-k} = \sigma(k, l)[(x_1 + C_1)^2 + \cdots + (x_l + C_l)^2], \]

where \( C_1, \ldots, C_l \) are arbitrary constants. Hence, according to c), we obtain the following set of solutions to equation (12) for \( l \leq n - 1 \):

\[ u^{1-k} = \sigma(k, l)[(x_1 + h_1(\omega))^2 + \cdots + (x_l + h_l(\omega))^2], \quad k \neq \frac{l}{l-2}, \]

where \( \omega \) is an arbitrary solution of system (10) and \( h_1(\omega), \ldots, h_l(\omega) \) are arbitrary twice differentiable functions of \( \omega \). In particular, if \( n = 3 \) and \( l = 1 \), then equation (12) possesses in the space \( \mathbb{R}_{1,3} \) the solution set

\[ u^{1-k} = \frac{\lambda(1-k)^2}{2(1+k)}(x_1 + h(\omega))^2, \quad k \neq -1. \]

Next, let us consider the following solution of equation (1) [7]:

\[ u^{1-k} = \sigma(k, s)(x_0^2 - x_1^2 - \cdots - x_s^2), \quad s = 2, \ldots, n, \tag{14} \]

where

\[ \sigma(k, s) = -\frac{\lambda(1-k)^2}{2(s - ks + k + 1)}, \quad k \neq \frac{s+1}{s-1}. \]

Solution (14) defines the multiparameter solution set

\[ u^{1-k} = \sigma(k, s)[x_0^2 - x_1^2 - \cdots - x_s^2 - (x_{s+1} + C_{s+1})^2 - \cdots - (x_s + C_s)^2], \]

where \( C_{s+1}, \ldots, C_s \) are arbitrary constants. According to d) we obtain the following solution set for \( l \geq 2 \)

\[ u^{1-k} = \sigma(k, s)[x_0^2 - x_1^2 - \cdots - x_s^2 - (x_{s+1} + h_{s+1}(\omega))^2 - \cdots - (x_s + h_s(\omega))^2], \]

where \( \omega \) is an arbitrary solution of system (11), and \( h_{s+1}(\omega), \ldots, h_s(\omega) \) are arbitrary twice differentiable functions. In particular, if \( l = 2 \) and \( s = 3 \), then equation (1) possesses in the space \( \mathbb{R}_{1,3} \) the following solution set

\[ u^{1-k} = \frac{\lambda(1-k)^2}{4(k-2)}(x_0^2 - x_1^2 - x_2^2 - (x_3 - h_s(\omega))^2), \quad k \neq 2. \]

The equation

\[ \Box u + 6u^2 = 0 \tag{15} \]
possesses the solution \( u = \mathcal{P}(x_3 + C_2) \), where \( \mathcal{P}(x_3 + C_2) \) is an elliptic Weierstrass function with the invariants \( g_2 = 0 \) and \( g_3 = C_1 \). Therefore, according to c) we get the following set of solutions of equation (15):

\[
u = \mathcal{P}(x_3 + h(\omega)) ,
\]

where \( \omega \) is an arbitrary solution to system (10) and \( h(\omega) \) is an arbitrary twice differentiable function of \( \omega \).

Next consider the Liouville equation

\[\Box u + \lambda \exp u = 0.\] (16)

The symmetry ansatz \( u = \varphi(\omega_1), \omega_1 = x_3 \), reduces equation (16) to the equation

\[\frac{d^2\varphi}{d\omega_1^2} = \lambda \exp \varphi(\omega_1).\]

Integrating this equation, we obtain that \( \varphi \) coincides with one of the following functions:

\[
\ln \left\{ -\frac{C_1}{2\lambda} \sec^2 \left[ \frac{\sqrt{-C_1}}{2} (\omega_1 + C_2) \right] \right\} \quad (C_1 < 0, \lambda > 0, C_2 \in \mathbb{R});
\]

\[
\ln \left\{ \frac{2C_1 C_2 \exp(\sqrt{C_1} \omega_1)}{\lambda[1 - C_2 \exp(\sqrt{C_1} \omega_1)]^2} \right\} \quad (C_1 > 0, \lambda C_2 > 0);
\]

\[- \ln \left( \frac{\lambda}{2} \omega_1 + C \right)^2 .
\]

Hence, according to c) we get the following solutions set for equation (16):

\[
u = \ln \left\{ -\frac{h_1(\omega)}{2\lambda} \sec^2 \left[ \frac{\sqrt{-h_1(\omega)}}{2} (\omega_1 + h_2(\omega)) \right] \right\} \quad (h_1(\omega) < 0, \lambda > 0); \]

\[
u = \ln \left\{ \frac{2h_1(\omega) h_2(\omega) \exp(\sqrt{h_1(\omega)} \omega_1)}{\lambda[1 - h_2(\omega) \exp(\sqrt{h_1(\omega)} \omega_1)]^2} \right\} \quad (h_1(\omega) > 0, \lambda h_2(\omega) > 0); \]

\[
u = - \ln \left( \frac{\lambda}{2} \omega_1 + h(\omega) \right)^2 ,
\]

where \( h_1(\omega), h_2(\omega), h(\omega) \) are arbitrary twice differentiable functions; \( \omega \) is an arbitrary solution to system (10).

Using, for example, the solution to the Liouville equation (16) \[7\]

\[
u = \ln \frac{2(s - 2)}{\lambda[x_1^2 - x_1^2 - \cdots - x_s^2]} , \quad s \neq 2,
\]

we obtain the wide class of solutions to the Liouville equation

\[
u = \ln \frac{2(s - 2)}{\lambda[x_1^2 - x_1^2 - \cdots - x_s^2 - (x_{i+1} + h_{i+1}(\omega))^2 - \cdots - (x_s + h_s(\omega))^2]} ,
\]
where \( \omega \) is an arbitrary solution to system (11), and \( h_1(\omega), \ldots, h_s(\omega) \) are arbitrary twice differentiable functions. If \( s = 3 \), then equation (16) possesses in the space \( \mathbb{R}_{1,3} \) the following solution set

\[
u = \ln \left[ \frac{2}{\lambda [x^2_0 - x^2_1 - x^2_2 - (x_3 + h_3(\omega))^2]} \right].
\]

Let us consider now the sine-Gordon equation

\[ \square u + \sin u = 0. \]

Doing in an analogous way, we get the following solutions:

\[
u = 4 \arctan h_1(\omega) e^{\alpha x_3} - \frac{1}{2} (1 - \epsilon) \pi, \quad \epsilon_0 = \pm 1, \quad \epsilon = \pm 1;
\]

\[
u = 2 \arccos [dn(x_3 + h_1(\omega)), m] + \frac{1}{2} (1 + \epsilon) \pi, \quad 0 < m < 1;
\]

\[
u = 2 \arccos \left[ cn \left( \frac{x_3 + h_1(\omega)}{m} \right), m \right] + \frac{1}{2} (1 + \epsilon) \pi, \quad 0 < m < 1,
\]

where \( h_1(\omega) \) is an arbitrary twice differentiable function, \( \omega \) is an arbitrary solution to system (10).


