On exact solutions of an equation of nonlinear acoustics

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New extended classes of exact solutions of the multidimensional nonlinear acoustic equation \( u_{00} = u \Delta u \) are obtained.

A lot of equations of nonlinear acoustics, theory of nonlinear waves are equations of the form

\[
u_{00} = c \left( \vec{x}, u, u_1 \right) \Delta u,
\]

where \( u = u(\vec{x}) \), \( \vec{x} = (x_0, x_1, \ldots, x_n) \in \mathbb{R}_{1,n} \); \( c \left( \vec{x}, u, u_1 \right) \) is an arbitrary differentiable function,

\[
\Delta u = \frac{\partial^2 u}{\partial x_1^2} + \cdots + \frac{\partial^2 u}{\partial x_n^2}, \quad u_{00} = \frac{\partial^2 u}{\partial x_0^2},
\]

\( u \) is the set of all possible derivatives of the second order. Group properties of equation (1) were studied in [1].

If \( c \left( \vec{x}, u, u_1 \right) = u \), then equation (1) takes the form

\[
u_{00} = u \Delta u.
\]

P. Olver and Ph. Rosenau [2] constructed solutions of the one-dimensional equation of acoustics (2), that can’t be obtained by using S. Lie’s method. In paper [3], the conditional symmetry of equation (2) was investigated. Under the conditional symmetry we mean the symmetry of some subset of solutions of the given equation. In [1, 3], 12 types of
nonequivalent conditional symmetry operators of equation (2) we found, with the help of which wide classes of exact solutions of the given equation were constructed. Note that in many cases ansatzes corresponding to conditional symmetry operators reduce the initial nonlinear equation to linear one.

In the present paper, proceeding from reflections different from the conception of conditional symmetry, we constructed wider classes of exact solutions of equation (2), than in [1,2]. In constructing these solutions, we essentially used solutions with separated variables [4]. It’s worth noting that in many cases to construct solutions with separated variables is essentially easier than to obtain conditional symmetry operators.

1. We look for solutions of equation (2) in the form

\[ u = a(x_0) b(x), \]

where functions \( a(x_0) \) and \( b(x) \) differ from constants. Substituting this equation (2) we get

\[ a''b = a^2b A \]

Here \( a'' \) means the second derivative of the function \( a(x_0) \) with respect to variable \( x_0 \). It follows from the latter equality that functions \( a'' \) and \( a^2 \) are linearly dependent, i.e. \( a'' = \alpha a^2 \) for some \( \alpha \in \mathbb{R} \). Also we obtain \( A b = \alpha \). If \( a = \mu x_0 + \nu \), then \( \alpha = 0 \), and a solution of equation (2) is of the form

\[ u = (\mu x_0 + \nu) b(x), \quad (3) \]

where \( \Delta b = 0 \).

If \( \mu \neq 0 \), then setting \( a_1 = \frac{\alpha}{\mu} a, b_1 = \frac{6}{\alpha} b \), we get \( a''_1 = 6a_1^2, \Delta b_1 = 6 \). Therefore, in the case \( \alpha \neq 0 \) solutions of equation (2) are of the form

\[ u = \left[ \frac{3}{k} (x_1^2 + \cdots + x_n^2) + G_\alpha(x_1, \ldots, x_n) \right] \phi(x_0), \quad (4) \]

\[ u = \left[ \frac{3}{k} (x_1^2 + \cdots + x_n^2) + G_\alpha(x_1, \ldots, x_n) \right] x_0^{-2}, \quad (5) \]

where \( 1 \leq k \leq n; \phi(x_0) \) is \( \Phi \) Weierstrass function with the invariants \( g_2 = 0, g_3 = C_1 \), \( \Delta G_\alpha = 0 \).

2. The solution

\[ u = (\mu x_0 + \nu) b(x) + b_1(x), \quad (6) \]

is a generalization of solution (3), where \( \Delta b = 0, \Delta b_1 = 0 \).
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Solution (4) is a partial case of the more general solution

\[ u = \phi(x)p(x_0) + G(x_0, x), \]

where

\[ \phi(x) = \frac{3}{k} (x_1^2 + \cdots + x_k^2) + G_\alpha(x_1, \ldots, x_n). \]

Substituting into equation (2) we find

\[ v_{00} = \phi(x)p(x_0)\Delta v + 6p(x_0)v + v\Delta v. \tag{7} \]

If function \( v \) depends only on \( x_0 \), then \( v_{00} = 6p(x_0)v \) and we have the following solution of equation (2):

\[ u = \left[ \frac{3}{k} (x_1^2 + \cdots + x_k^2) + G_\alpha(x_1, \ldots, x_n) \right] p(x_0) + f(x_0), \tag{8} \]

where \( f(x_0) \) is Lamé function [5].

If function \( v \) depends on \( x_0 \) and \( x \), then we look for a solution of equation (7) in the form \( v = a(x_0)b(x) \), where \( a(x_0) \) and \( b(x) \) differ from constants. Substituting into equation (7) we get

\[ a''b = a(x_0)p(x_0)\phi \Delta b + 6a(x_0)p(x_0)b + a^2(x_0)b \Delta b. \tag{9} \]

Equality (9) means functions \( a'' \), \( ap \) and \( a^2 \) are linearly dependent. If we assume \( a^2 = ap \), \( a \in \mathbb{R} \) or \( a = ap \) and the solution we are looking for can be presented in the form \( u = \phi(x_0)d(x) \), i.e. we are under conditions of p.l. Hence, we can suppose functions \( ap \) and \( a^2 \) are linearly independent. Therefore,

\[ a'' = ap^2 + \beta ap. \tag{10} \]

Substituting into (9) we come to

\[ (ab - b\Delta b)a^2 + (\beta b - \phi\Delta b - 6b)ap = 0. \]

Since \( a^2 \) and \( ap \) are linearly independent,

\[ ab - b\Delta b = 0, \quad \beta b - \phi\Delta b - 6b = 0. \tag{11} \]

From system (11) it follows

\[ \Delta b = \alpha, \quad \alpha \phi = (\beta - 6)b. \]
If $a = 0$, then $\beta = 6$ and we obtain the following exact solution of equation (2):

$$u = \left[ \frac{3}{k} \left( x_1^2 + \cdots + x_k^2 \right) + G_\alpha(x_1, \ldots, x_n) \right] \varphi(x_0) +$$
$$+ \Phi_\alpha(x_1, \ldots, x_n) f(x_0), \quad (12)$$

where $1 \leq k \leq n$, $\Delta G_\alpha = 0$, $\Delta \Phi_\alpha = 0$, $f'' = 6p f$ is $\frac{1}{6}$ Lamé function.

If $\alpha \neq 0$, then one can assume $\alpha = 1$. For this reason $\varphi = (\beta - 6) b$, whence $\Delta \varphi = (\beta - 6) \Delta b$, i.e. $\beta = 12$. This means $b = \frac{1}{6} \varphi$ and we are under conditions of p.1.

The solution

$$u = \left[ \frac{3}{k} \left( x_1^2 + \cdots + x_k^2 \right) + G_\alpha(x_1, \ldots, x_n) \right] x_0^{-2} +$$
$$+ \Phi_\alpha(x_1, \ldots, x_n) x_0^{-3}, \quad (13)$$

is a generalization of solution (5), where $1 \leq k \leq n$, $\Delta G_\alpha = 0$, $\Delta \Phi_\alpha = 0$.

3. Consider more complicated case, namely, we shall seek for a solution of equation (2) in the form

$$u = a(x_0) b(x) + c(x_0) d(x). \quad (14)$$

If functions $a(x_0)$ and $c(x_0)$ are linearly dependent, then $c(x_0) = \alpha a(x_0)$, $\alpha \in \mathbb{R}$ and therefore $u = a(x_0) b_1(x)$, where $b_1(x) = b(x) + \alpha d(x)$. This case was the subject of research in p.1. Hence, one can assume that functions $a(x_0)$ and $c(x_0)$ are linearly independent. For the same reason functions $b(x)$ and $d(x)$ are also linearly independent. Substituting (14) into equation (2) we come to

$$a''b + c''d = a^2(b \Delta b) + ac(b \Delta d) + ac(d \Delta b) + c^2(d \Delta d). \quad (15)$$

Equality (15) means functions $a^2$, $ac$, $c^2$, $a''$, $c''$ are linearly dependent. It is not easy to show that from linear independence of functions $a$ and $c$ it follows linear independence of functions $a^2$, $ac$ and $c^2$. In fact, if functions $a^2$, $ac$ and $c^2$ are linearly dependent, then $\alpha a^2 + \beta ac + \gamma c^2 = 0$ for some real numbers $\alpha$, $\beta$, $\gamma$ not being equal simultaneously zero. Assume $\alpha = 0$, then $\beta ac + \gamma c^2 = 0$, i.e. $c(\beta a + \gamma c) = 0$. From this it follows
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\[ \beta a + \gamma c = 0 \] and functions \( a \) and \( c \) are linearly dependent, that contradicts the hypothesis. Therefore, \( \alpha \neq 0 \) hence one can assume \( \alpha = 1 \). Since

\[ a^2 + \beta ac + \gamma c^2 = \left( a + \frac{\beta}{2} \right)^2 + \left( \gamma - \frac{\beta^2}{4} \right) c^2 = 0, \]

we have \( \gamma - \frac{\beta^2}{4} = -\delta^2 < 0 \). In consequence of this

\[ a^2 + \beta ac + \gamma c^2 = \left( a + \frac{\delta}{2} c - \delta c \right) \left( a + \frac{\delta}{2} c + \delta c \right) = 0. \]

From the latter equality we obtain \( a + \left( \frac{\beta}{2} - \delta \right) c = 0 \) or \( a + \left( \frac{\beta}{2} + \delta \right) c = 0 \). This means \( a \) and \( c \) are linearly dependent and we again come to contradiction.

Suppose function \( a^2, ac, c^2 \) and \( a'' \) also linearly independent. Then

\[ a'' = \alpha a^2 + \beta ac + \gamma c^2 + \delta a'' \quad (16) \]

for some \( \alpha, \beta, \gamma, \delta \in \mathbb{R} \). Substituting (16) into (15) we find coefficient of \( a'' \). It equals \( b + \delta d = 0 \), i.e. \( b \) and \( d \) are linearly dependent, that contradicts assumption. The contradiction obtained proves the system of functions \( a^2, ac, c^2 \) and \( a'' \) is linearly independent.

Let

\[ a'' = \alpha a^2 + \beta ac + \gamma c^2, \quad c'' = \alpha_1 a^2 + \beta_1 ac + \gamma_1 c^2, \quad (17) \]

where \( \alpha, \beta, \gamma, \alpha_1, \beta_1, \gamma_1 \in \mathbb{R} \). Substituting (17) into (15) and taking into account the linear independence of functions \( a^2, ac \) and \( c^2 \) we obtain

\[ \begin{align*}
\alpha b + \alpha_1 d &= b\Delta b, \\
\gamma b + \gamma_1 d &= d\Delta d, \\
\beta b + \beta_1 d &= b\Delta d + d\Delta b.
\end{align*} \quad (18) \]

Multiplying both parts of the first equation of system (18) by \( d^2 \), of the second equation - by \( b^2 \) and of the third one - by \( bd \), we get

\[ \beta b^2 d + \beta_1 bd^2 = \gamma b^3 + \gamma_1 b^2 d + \alpha bd^2 + \alpha_1 d^3. \quad (19) \]

From the linear independence of functions \( b \) and \( d \) it follows the linear independence of functions \( b^2 d, bd^2, b^3 \) and \( d^3 \). Therefore, from equality (19) we obtain \( \beta = \gamma_1, \beta_1 = \alpha, \gamma = 0, \alpha_1 = 0 \). Hence,

\[ a'' = \alpha a^2 + \beta ac, \quad c'' = \alpha a c + \beta c^2. \quad (20) \]
In system (20) \( \alpha \neq 0, \beta \neq 0 \). Denote \( c_1 = \beta c, d_1 = \frac{1}{\beta} d \), then

\[
\begin{align*}
\alpha'' &= \alpha a^2 + ac_1, \\
\Delta c_1 &= \alpha a c_1 + c_1^2, \\
\Delta d_1 &= 1.
\end{align*}
\]

This means one can take \( \beta = 1 \) in system (20). For this value of \( \beta \) system (20) has the following solution [5]:

\[
\alpha'' = \alpha^2(x_0) \left( \int \frac{dx_0}{a^2(x_0)} + \alpha \right), \\
c(x_0) = a(x_0) \int \frac{dx_0}{a^2(x_0)}.
\]

As a result, we obtain the following solution of equation (2):

\[
u = a(x_0)b(x) + d(x)a(x_0) \int \frac{dx_0}{a^2(x_0)},
\tag{21}
\]

where

\[
\begin{align*}
b(x) &= \frac{\alpha}{2k} (x_1^k + \cdots + x_n^k) + G_\alpha(x_1, \ldots, x_n), \\
d(x) &= \frac{1}{2l} (x_1^l + \cdots + x_n^l) + \Phi_\alpha(x_1, \ldots, x_n),
\end{align*}
\]

\(1 \leq k \leq n, 1 \leq l \leq n, \Delta G_\alpha = 0, \Delta \Phi_\alpha = 0\) and function \( a(x_0) \) is a solution of the equation

\[
\alpha'' = \alpha^2(x_0) \left( \int \frac{dx_0}{a^2(x_0)} + \alpha \right).
\]


