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RELATION BETWEEN THE MAXIMUM MODULUS AND THE MAXIMAL TERM OF DIRICHLET SERIES IN TERMS OF A CONVERGENCE CLASS

Summary

A relation between the growth of maximum modulus and the growth of maximal term of Dirichlet series in terms of a convergence class is investigated.

Keywords and phrases: Dirichlet series, convergence class

1. Introduction

Suppose that $\Lambda = (\lambda_n)$ is a sequence of positive numbers, increasing to $+\infty$, and $S(\Lambda, A)$ is the class of Dirichlet series

$$(1) \qquad F(s) = \sum_{n=0}^{\infty} a_n \exp\{s\lambda_n\}, \quad s = \sigma + it,$$

with abscissa of absolute convergence $\sigma_a = A \in (-\infty, +\infty]$. For $\sigma < A$ we put

$$M(\sigma, F) = \sup\{|F(\sigma + it)| : t \in \mathbb{R}\},$$

and let

$$\mu(\sigma, F) = \max\{|a_n| \exp(\sigma\lambda_n) : n \geq 0\}$$

be the maximal term of series (1),

$$\nu(\sigma, F) = \max\{n \geq 0 : |a_n| \exp(\sigma\lambda_n) = \mu(\sigma, F)\}$$

be its central index and

$$\varkappa_n = \frac{\ln|a_n| - \ln|a_{n+1}|}{\lambda_{n+1} - \lambda_n}.$$

By $\Omega(A)$ we denote the class of positive functions Φ, unbounded on $(-\infty, A)$, such that the derivative Φ' is continuously differentiable, positive and increasing to $+\infty$ on $(-\infty, A)$. For $\Phi \in \Omega(A)$ let φ be the function inverse to Φ' and let

$$\Psi(\sigma) = \sigma - \frac{\Phi(\sigma)}{\Phi'(\sigma)}$$

b on associated with Φ in the sense of Newton. Then [1; 2, p. 30] Ψ is continuously differentiable and increasing to A on $(-\infty, A)$ and φ is continuously differentiable and increasing to A on $(0, +\infty)$.

As in [3], we say that Dirichlet series belongs to a convergence Φ-class if

$$\int_{\sigma_0}^{A} \frac{\Phi'(\sigma)\ln M(\sigma, F)}{\Phi^2(\sigma)} d\sigma < +\infty. \tag{2}$$

By Cauchy inequality from (2) it follows that

$$\int_{\sigma_0}^{A} \frac{\Phi'(\sigma)\ln \mu(\sigma, F)}{\Phi^2(\sigma)} d\sigma < +\infty. \tag{3}$$

The next question arises: which conditions do ensure that (3) implies (2)? Further we assume that either $A = +\infty$ or $A = 0$, because the case $A \in (-\infty, +\infty)$ can be reduced to the case $A = 0$ by substituting $s - A$ for s.

In [4] it is proved that if

$$\int_{\sigma_0}^{\infty} \frac{\Phi'(\sigma)\ln \Phi'(\sigma)}{\Phi^2(\sigma)} d\sigma < +\infty, \tag{4}$$

then for each entire ($A = +\infty$) Dirichlet series with $\lambda_n = n$ relations (2) and (3) are equivalent. We remark that from (4) it follows that $\ln \Phi'(\sigma) = o(\Phi(\sigma))$ as $\sigma \to +\infty$.

If $\Phi(\sigma) = e^{\varrho\sigma}$ ($\varrho > 0$) then from (2) we obtain the definition of a convergence class, introduced in [5]. For such convergence class in [6] the following theorem is proved.

Theorem A. *For the relations*

$$\int_0^{\infty} \frac{\ln M(\sigma, F)}{\exp\{\varrho\sigma\}} d\sigma < +\infty \quad \text{and} \quad \int_0^{\infty} \frac{\ln \mu(\sigma, F)}{\exp\{\varrho\sigma\}} d\sigma < +\infty$$

to be equivalent for each $F \in S(\Lambda, +\infty)$ it is necessary and sufficient that

$$\ln n = O(\lambda_n) \qquad (n \to \infty).$$

If $\Phi(\sigma) = \sigma^p$ $(p > 1)$ for $\sigma \geq \sigma_0$ then from (2) we obtain the definition of a logarithmic convergence class, for which the following theorem is true [7].

Theorem B. *For the relations*

$$\int_1^\infty \sigma^{-(p+1)} \ln M(\sigma, F) d\sigma < +\infty \quad \text{and} \quad \int_1^\infty \sigma^{-(p+1)} \ln \mu(\sigma, F) d\sigma < +\infty$$

to be equivalent for each $F \in S(\Lambda, +\infty)$ it is necessary and sufficient that

$$\ln n = O(\lambda_n^{p/(p-1)}) \qquad (n \to \infty).$$

For Dirichlet series with $\sigma_a = 0$ the convergence class is defined [8] by condition (2) with $\Phi(\sigma) = |\sigma|^{-\varrho}$ $(\varrho > 0)$. The following theorem is true.

Theorem C. *For the relations*

$$\int_{-1}^0 |\sigma|^{\varrho-1} \ln M(\sigma, F) d\sigma < +\infty \quad \text{and} \quad \int_{-1}^0 |\sigma|^{\varrho-1} \ln \mu(\sigma, F) d\sigma < +\infty$$

to be equivalent for each $F \in S(\Lambda, 0)$ it is necessary and sufficient that

$$\ln \ln n = o(\ln \lambda_n) \qquad (n \to \infty).$$

Finally, if we choose in (2) $\Phi(\sigma) = e^{\varrho/|\sigma|}$ $(\varrho > 0)$ then we obtain [8] the definition of the convergence class for Dirichlet series with $\sigma_a = 0$ of finite R-order. In [9] the following theorem is proved.

Theorem D. *For the relations*

$$\int_{-1}^0 \frac{\ln M(\sigma, F)}{|\sigma|^2 \exp\{\varrho/|\sigma|\}} d\sigma < +\infty \quad \text{and} \quad \int_{-1}^0 \frac{\ln \mu(\sigma, F)}{|\sigma|^2 \exp\{\varrho/|\sigma|\}} d\sigma < +\infty$$

to be equivalent for each $F \in S(\Lambda, 0)$ it is necessary that

$$\ln n = O(\lambda_n / \ln^2 \lambda_n) \qquad (n \to \infty)$$

and sufficient that

$$\ln n = O(\lambda_n / \ln^q \lambda_n) \qquad (n \to \infty) \quad \text{with} \quad q > 3.$$

The aim of the present investigation is to find condition on (λ_n), under which the relations (2) and (3) are equivalent in the case when the function Φ increases rapidly enough, that is $\Phi'(\sigma)/\Phi(\sigma)$ is a nondecreasing function.

2. Sufficient condition

Let $n(t) = \sum_{\lambda_n \le t} 1$ be the counting function of the sequence (λ_n). From the proof of Theorem 1 from [10] the following statement follows.

Lemma 1. *Let either $A = +\infty$ or $A = 0$, $\Phi \in \Omega(A)$, and $\Phi'(\sigma)/\Phi(\sigma)$ be a function, nondecreasing on $[\sigma_0, A)$. If $\ln \mu(\sigma, F) \le \Phi(\sigma)$ for all $\sigma \in [\sigma_0, A)$ and $\ln n(t) = o(t)$ as $t \to +\infty$, then $M(\sigma, F) \le \mu(\sigma, F) n(\gamma(\sigma)) + 1$ for $\sigma \in [\sigma_0, A)$, where $\gamma(\sigma) = \Phi'(\Psi^{-1}(\sigma + \beta(\sigma)))$ and $\beta(\sigma) = \Phi(\sigma)/\Phi'(\Psi^{-1}(\sigma))$.*

Using Lemma 1 we prove the following theorem.

Theorem 1. *Let either $A = +\infty$ or $A = 0$, $\Phi \in \Omega(A)$, $\Phi'(\sigma)/\Phi(\sigma)$ be a function, nondecreasing on $[\sigma_0, A)$, and*

$$\frac{\Phi''(\sigma)\Phi(\sigma)}{(\Phi'(\sigma))^2} \le H < +\infty$$

for all $\sigma \in [\sigma_0, A)$. Then, for conditions (2) and (3) to be equivalent for any function $f \in S(\Lambda, A)$ it is sufficient that

$$\int_{t_0}^{\infty} \frac{\ln n(t)}{t\Phi(\Psi(\varphi(t)))} dt < +\infty. \tag{5}$$

Proof. Since

$$\Psi'(\sigma) = \frac{\Phi''(\sigma)\Phi(\sigma)}{(\Phi'(\sigma))^2} \quad \text{and} \quad \frac{\Phi(\Psi(\varphi(t)))}{\Phi'(\Psi(\varphi(t)))} \ge \frac{\Phi(\varphi(t))}{\Phi'(\varphi(t))}$$

we have

$$\begin{aligned}
&\varliminf_{t \to +\infty} \Phi(\Psi(\varphi(t))) \int_t^{\infty} \frac{dx}{x\Phi(\Psi(\varphi(x)))} \\
&\ge \varliminf_{t \to +\infty} \frac{(\Phi(\Psi(\varphi(t))))^2}{t\Phi(\Psi(\varphi(t)))\Phi'(\Psi(\varphi(t)))\Psi'(\varphi(t))\varphi'(t)} = \\
&= \varliminf_{t \to +\infty} \frac{\Phi(\Psi(\varphi(t)))}{\Phi'(\Psi(\varphi(t)))} \frac{(\Phi'(\varphi(t)))^2}{\Phi''(\varphi(t))\Phi(\varphi(t))t\varphi'(t)} \\
&\ge \varliminf_{t \to +\infty} \frac{\Phi(\varphi(t))}{\Phi'(\varphi(t))} \frac{(\Phi'(\varphi(t)))^2}{\Phi''(\varphi(t))\Phi(\varphi(t))t\varphi'(t)} = 1.
\end{aligned}$$

Therefore, for every $\varepsilon > 0$ and all $t \ge t_0(\varepsilon)$ we obtain from (5)

$$\begin{aligned}
\varepsilon > \int_t^{\infty} \frac{\ln n(x)}{x\Phi(\Psi(\varphi(x)))} dx &\ge \ln n(t) \int_t^{\infty} \frac{dx}{x\Phi(\Psi(\varphi(x)))} \\
&\ge \frac{(1 + o(1)) \ln n(t)}{\Phi(\Psi(\varphi(t)))}, \quad t \to +\infty,
\end{aligned}$$

whence it follows that

$$\ln n(t) = o(\Phi(\Psi(\varphi(t)))) \quad \text{as} \quad t \to +\infty.$$

But $\Phi(\Psi(\varphi(t))) \le \Phi(\varphi(t)) = O(t)$ as $t \to +\infty$, because $\Phi(\sigma) = O(\Phi'(\sigma))$ as $\sigma \uparrow A$. Thus, $\ln n(t) = o(t)$ as $t \to +\infty$.

From (3) it follows that $\ln \mu(\sigma, F) \le \Phi(\sigma)$ for all $\sigma \in [\sigma_0, A)$. Therefore, by Lemma 1 $\ln M(\sigma, F) \le \ln \mu(\sigma, F) + \ln n(\gamma(\sigma)) + o(1)$ as $\sigma \uparrow A$. Hence it follows that (3) implies (2) provided

$$I(\Lambda) = \int_{\sigma_0}^{A} \frac{\Phi'(\sigma) \ln n(\Phi'(\Psi^{-1}(\sigma + \beta(\sigma)))}{\Phi^2(\sigma)} d\sigma < +\infty. \tag{6}$$

From the nondecrease of Φ'/Φ it follows [10] that $\sigma + \beta(\sigma) \le \Psi^{-1}(\sigma)$ and $\Phi(\sigma) \le e\Phi(\Psi(\sigma))$ for $\sigma_0 \le \sigma < A$. Therefore,

$$\begin{aligned}
I(\Lambda) &\le \int_{\sigma_0}^{A} \frac{\Phi'(\sigma)}{\Phi(\sigma)} \frac{\ln n(\Phi'(\Psi^{-1}(\Psi^{-1}(\sigma))))}{\Phi(\sigma)} d\sigma \\
&\le e \int_{\sigma_0}^{A} \frac{\Phi'(\Psi^{-1}(\sigma))}{\Phi(\Psi^{-1}(\sigma))} \frac{\ln n(\Phi'(\Psi^{-1}(\Psi^{-1}(\sigma))))}{\Phi(\Psi^{-1}(\sigma))} d\sigma = \\
&= e \int_{\sigma_0}^{A} \frac{\Phi'(\Psi^{-1}(\sigma))}{\Phi(\Psi^{-1}(\sigma))} \frac{\ln n(\Phi'(\Psi^{-1}(\Psi^{-1}(\sigma))))}{\Phi(\Psi^{-1}(\sigma))} \frac{d\Psi^{-1}(\sigma)}{(\Psi^{-1}(\sigma))'} = \\
&= e \int_{\Psi^{-1}(\sigma_0)}^{A} \frac{\Phi'(\sigma)}{\Phi(\sigma)} \frac{\ln n(\Phi'(\Psi^{-1}(\sigma)))}{\Phi(\sigma)} \Psi'(\sigma) d\sigma \\
&\le eH \int_{\Psi^{-1}(\sigma_0)}^{A} \frac{\Phi'(\sigma)}{\Phi(\sigma)} \frac{\ln n(\Phi'(\Psi^{-1}(\sigma)))}{\Phi(\sigma)} d\sigma.
\end{aligned}$$

Since

$$(\Psi(\varphi(x)))' = \frac{\Phi(\varphi(x))}{x^2},$$

hence and from (5) we obtain

$$\begin{aligned}
I(\Lambda) &\le eH \int_{x_0}^{\infty} \frac{\Phi'(\Psi(\varphi(x)))}{\Phi(\Psi(\varphi(x)))} \frac{\ln n(x)}{\Phi(\Psi(\varphi(x)))} \frac{\Phi(\varphi(x))}{x^2} dx \le \\
&\le eH \int_{x_0}^{\infty} \frac{\Phi'(\varphi(x))}{\Phi(\varphi(x))} \frac{\ln n(x)}{\Phi(\Psi(\varphi(x)))} \frac{\Phi(\varphi(x))}{x^2} dx = \int_{x_0}^{\infty} \frac{\ln n(x)}{x\Phi(\Psi(\varphi(x)))} dx < +\infty,
\end{aligned}$$

that is (6) holds, and Theorem 1 is proved.

3. Necessary condition

We need the following lemmas.

Lemma 2. [10, 11] *Suppose that γ, defined on $[0,+\infty)$, is a positive, continuous and increasing to $+\infty$ function and*

$$\varliminf_{n\to+\infty}\frac{\ln n}{\gamma(\lambda_n)}>1.$$

Then there exists a subsequence (λ_k^*) of the sequence (λ_n) such that

$$k\le\exp\{\gamma(\lambda_k^*)\}+1\quad\text{for all}\quad k\ge1\quad\text{and}\quad k_j\ge\exp\{\gamma(\lambda_{k_j}^*)\}$$

for an increasing sequence of (k_j) of positive integers.

Lemma 3. [3] *The relation* (3) *holds if and only if*

$$\int\limits_{\sigma_0}^{A}\frac{\lambda_{\nu(\sigma,F)}}{\Phi(\sigma)}d\sigma<+\infty. \tag{7}$$

Lemma 4. [12, p. 115] *If $\ln n=o(\lambda_n)$ as $n\to\infty$ then*

$$\sigma_a=-\varlimsup_{n\to+\infty}\frac{\ln|a_n|}{\lambda_n}.$$

Using Lemmas 2–4, we prove the following theorem.

Theorem 2. *Suppose that $A=+\infty$ or $A=0$ and the function $\Phi\in\Omega(A)$ is such that*

$$\frac{\Phi'(\sigma)}{\Phi(\sigma)}\uparrow+\infty,\quad\frac{\Phi'(\sigma)}{\Phi^2(\sigma)}\downarrow0$$

as $\sigma\uparrow A$ and

$$\Phi(\varphi(x))\Phi'(\Phi^{-1}(x))=O(x^2)\quad\text{as}\quad x\to+\infty.$$

Then, for relations (2) *and* (3) *to be equivalent for any function $F\in S(\Lambda,A)$, it is necessary that*

$$\ln n=O\left(\frac{\lambda_n^2}{\Phi'(\Phi^{-1}(\lambda_n))}\right),\quad n\to\infty. \tag{8}$$

Proof. At first we note that

$$\frac{x^2}{\Phi'(\Phi^{-1}(x))}\uparrow+\infty$$

and

$$\frac{x}{\Phi'(\Phi^{-1}(x))}\downarrow0\quad\text{as}\quad x\to+\infty.$$

Now we assume that the sequence Λ does not satisfy condition (8). Then there exists defined on $(0, +\infty)$, positive, continuous and slowly increasing to $+\infty$ function l such that

$$\frac{xl(x)}{\Phi'(\Phi^{-1}(x))} \to 0, \quad x \to +\infty, \tag{9}$$

and

$$\varlimsup_{n\to+\infty} \frac{\Phi'(\Phi^{-1}(\lambda_n)) \ln n}{\lambda_n^2 l(\lambda_n)} > 1. \tag{10}$$

Conditions of Theorem 2 imply

$$\frac{1}{x}\Phi'\left(\Phi^{-1}\left(\frac{x^2 l(x)}{\Phi'(\Phi^{-1}(x))}\right)\right) \to +\infty$$

as $x \to +\infty$. Indeed, if

$$\Phi'\left(\Phi^{-1}\left(\frac{x_k^2 l(x_k)}{\Phi'(\Phi^{-1}(x_k))}\right)\right) \le K_0 x_k$$

for some $K_0 = \text{const} \ge 1$ and a sequence (x_k) increasing to $+\infty$ then

$$l(x_k) \le K_0 \frac{\Phi(\varphi(K_0 x_k))}{K_0 x_k} \frac{\Phi'(\Phi^{-1}(x_k))}{x_k} \le K_0 \frac{\Phi(\varphi(x_k))\Phi'(\Phi^{-1}(x_k))}{x_k^2} = O(1), \ k \to \infty,$$

which is impossible. Hence it follows that there exists defined on $(0, +\infty)$, positive, continuous and increasing to $+\infty$ function L such that

$$\Phi'\left(\Phi^{-1}\left(\frac{x^2 l(x)}{\Phi'(\Phi^{-1}(x))}\right)\right) \ge xL(x), \quad x \ge x_0. \tag{11}$$

In view of (10) by Lemma 2 there exists a subsequence (λ_k^*) of the sequence (λ_n) such that

$$k \le \exp\left\{\frac{(\lambda_k^*)^2 l(\lambda_k^*)}{\Phi'(\Phi^{-1}(\lambda_k^*))}\right\} + 1$$

for all $k \ge 1$ and

$$k_j \ge \exp\left\{\frac{(\lambda_{k_j}^*)^2 l(\lambda_{k_j}^*)}{\Phi'(\Phi^{-1}(\lambda_{k_j}^*))}\right\}$$

for an increasing sequence of (k_j) of positive integers.

For $\lambda_n \notin (\lambda_k^*)$ we put $a_n = 0$ and in the obtained Dirichlet series we replace λ_k^* by λ_n. We come to Dirichlet series (1) with the exponents λ_n satisfying following conditions

$$\ln n \le \frac{\lambda_n^2 l(\lambda_n)}{\Phi'(\Phi^{-1}(\lambda_n))} + 1, \quad n \ge 1, \tag{12}$$

and

$$\ln n_j \ge \frac{\lambda_{n_j}^2 l(\lambda_{n_j})}{\Phi'(\Phi^{-1}(\lambda_{n_j}))} \tag{13}$$

for an increasing sequence of (n_j) of positive integers. We can consider that the sequence (n_j) is such that $n_{j+1} > 2n_j$ and $\lambda_{m_j} > 2\lambda_{n_j}$ for $m_j = [n_{j+1}/2]$ and all $j \geq 1$, and

$$\sum_{j=1}^{\infty} \frac{1}{L(\lambda_{n_{j+1}})} < +\infty. \tag{14}$$

Let (q_k) be a sequence, increasing to A. We put $n_0 = 0$, $a_{n_0} = 1$, $a_n = 0$ for all $n_j < n < m_j$,

$$a_{n_{j+1}} = \prod_{k=0}^{j} \exp\{-q_k(\lambda_{n_{k+1}} - \lambda_{n_k})\}, \quad j = 0, 1, \dots, \tag{15}$$

and

$$a_n = a_{n_j} \exp\{-q_j(\lambda_n - \lambda_{n_j})\}, \quad m_j \leq n < n_{j+1}, \tag{16}$$

that is we obtain the Dirichlet series

$$F^*(s) = \sum_{j=0}^{\infty} \left(a_{n_j} \exp\{s\lambda_{n_j}\} + \sum_{n=m_j}^{n_{j+1}-1} a_n \exp\{s\lambda_n\} \right). \tag{17}$$

From (15) and (16) it follows that

$$\frac{\ln a_{n_j} - \ln a_{n_{j+1}}}{\lambda_{n_{j+1}} - \lambda_{n_j}} = \frac{\ln a_{n_j} - \ln a_{m_j}}{\lambda_{m_j} - \lambda_{n_j}} = \frac{\ln a_{n_j} - \ln a_{n_j+1}}{\lambda_{n_j+1} - \lambda_{n_j}} =$$
$$= \frac{\ln a_n - \ln a_{n+1}}{\lambda_{n+1} - \lambda_n} = q_j, \quad m_j \leq n < n_{j+1} - 1.$$

Therefore, if $q_j \leq \sigma < q_{j+1}$ then $\nu(\sigma, F^*) = n_{j+1}$ and $\mu(\sigma, F^*) = a_{n_{j+1}} \exp\{\sigma\lambda_{n_{j+1}}\}$. Since

$$\Phi''(\sigma)\Phi(\sigma) - (\Phi'(\sigma))^2 = \Phi^2(\sigma)(\Phi'(\sigma)/\Phi(\sigma))' \geq 0,$$

hence we have

$$\int_{q_1}^{A} \frac{\lambda_{\nu(\sigma,F^*)}}{\Phi(\sigma)} d\sigma = \sum_{j=1}^{\infty} \int_{q_j}^{q_{j+1}} \frac{\lambda_{\nu(\sigma,F^*)}}{\Phi(\sigma)} d\sigma = \sum_{j=1}^{\infty} \lambda_{n_{j+1}} \int_{q_j}^{q_{j+1}} \frac{d\sigma}{\Phi(\sigma)} \leq$$
$$\leq \sum_{j=1}^{\infty} \lambda_{n_{j+1}} \int_{q_j}^{q_{j+1}} \frac{\Phi''(\sigma) d\sigma}{(\Phi'(\sigma))^2} \leq \sum_{j=1}^{\infty} \frac{\lambda_{n_{j+1}}}{\Phi'(q_j)}. \tag{18}$$

On the other hand,

$$M(q_j, F^*) \geq \sum_{n=m_j}^{n_{j+1}-1} a_n \exp\{q_j\lambda_n\} = (n_{j+1} - m_j)\mu(q_j, F^*) \geq K_1 n_{j+1}, \tag{19}$$

where $K_1 \equiv \text{const}$.

We choose

$$q_j = \Phi^{-1}\left(\frac{\lambda_{n_{j+1}}^2 l(\lambda_{n_{j+1}})}{\Phi'(\Phi^{-1}(\lambda_{n_{j+1}}))} \right).$$

Then from (19) and (13) we obtain

$$\ln M(q_j, F^*) \geq \ln n_{j+1} + \ln K_1 \geq \frac{\lambda_{n_{j+1}}^2 l(\lambda_{n_{j+1}})}{\Phi'(\Phi^{-1}(\lambda_{n_{j+1}}))} + \ln K_1 = \Phi(q_j) + \ln K_1,$$

that is the relation (2) does not hold, because (2) implies $\ln M(\sigma, F) = o(\Phi(\sigma))$, $\sigma \uparrow A$.

From (18), (11) and (14) we have

$$\int_{q_1}^{A} \frac{\lambda_{\nu(\sigma,F^*)}}{\Phi(\sigma)} d\sigma \leq \sum_{j=1}^{\infty} \frac{\lambda_{n_{j+1}}}{\Phi'\left(\Phi^{-1}\left(\frac{\lambda_{n_{j+1}}^2 l(\lambda_{n_{j+1}})}{\Phi'(\Phi^{-1}(\lambda_{n_{j+1}}))}\right)\right)} \leq \sum_{j=1}^{\infty} \frac{1}{L(\lambda_{n_{j+1}})} < +\infty,$$

that is by Lemma 3 relation (3) holds.

Finally, we prove that $\sigma_a = A$. Since $q_k \uparrow A$ $(k \to \infty)$, from (15) we have

$$\frac{\ln a_{n_{j+1}}}{\lambda_{n_{j+1}}} = \frac{-\sum\limits_{k=0}^{j} q_k(\lambda_{n_{k+1}} - \lambda_{n_k})}{\sum\limits_{k=0}^{j} (\lambda_{n_{k+1}} - \lambda_{n_k})} \downarrow -A, \quad j \to \infty.$$

If $m_j \leq n < n_{j+1}$ and $A = 0$ then from (16) we obtain

$$\frac{\ln a_n}{\lambda_n} = \frac{\ln a_{n_j}}{\lambda_{n_j}} \frac{\lambda_{n_j}}{\lambda_n} - q_j + q_j \frac{\lambda_{n_j}}{\lambda_n} = o\left(\frac{\lambda_{n_j}}{\lambda_n}\right) - q_j \left(1 - \frac{\lambda_{n_j}}{\lambda_n}\right) = o(1), \quad n \to \infty,$$

and if $A = +\infty$ then

$$\frac{\ln a_n}{\lambda_n} \leq -q_j \left(1 - \frac{\lambda_{n_j}}{\lambda_n}\right) \leq -\frac{q_j}{2} \to -\infty, \quad j \to \infty.$$

Therefore,

$$\varliminf_{n \to +\infty} \frac{\ln a_n}{\lambda_n} = -A,$$

and since, in view of (12) and (9), $\ln n = o(\lambda_n)$ $(n \to \infty)$ by Lemma 4 we have $\sigma_a = A$.

Thus, if the sequence Λ does not satisfy condition (8) then there exists a function $F \in S(\Lambda, A)$, for which relation (2) and (3) are not equivalent. Theorem 2 is proved.

4. Remarks

Using Theorems 1 and 2, it is easy to prove the following statement, which is a slight generalization of Theorem D.

Corollary 1. *Let $\varrho > 0$. For the relations*

$$\int_{-1}^{0} \frac{\ln M(\sigma, F)}{|\sigma|^2 \exp\{\varrho/|\sigma|\}} d\sigma < +\infty \quad \textit{and} \quad \int_{-1}^{0} \frac{\ln \mu(\sigma, F)}{|\sigma|^2 \exp\{\varrho/|\sigma|\}} d\sigma < +\infty$$

to be equivalent for any function $f \in S(\Lambda, 0)$, *it is sufficient that*

$$\int_{t_0}^{\infty} \frac{\ln^2 t}{t^2} \ln n(t)dt < +\infty \tag{20}$$

and necessary that $\ln n(t) = O(t \ln^{-2} t)$ $(t \to +\infty)$.

Indeed, for

$$\Phi(\sigma) = \exp\left\{\frac{\varrho}{|\sigma|}\right\}$$

we have

$$\Phi'(\sigma) = \frac{\varrho}{|\sigma|^2} \exp\left\{\frac{\varrho}{|\sigma|}\right\},$$

$$|\Phi^{-1}(x)| = \frac{\varrho}{\ln x}, \quad \Phi'(\Phi^{-1}(x)) = \frac{x \ln^2 x}{\varrho}$$

and since

$$\frac{\varrho}{|\varphi(x)|^2} \exp\left\{\frac{\varrho}{|\varphi(x)|}\right\} \equiv x$$

we also have

$$|\varphi(x)| = \frac{(1 + o(1))\varrho}{\ln x}$$

and

$$\Phi(\varphi(x)) = \frac{x|\varphi(x)|^2}{\varrho} = \frac{(1 + o(1))x\varrho}{\ln^2 x} \quad \text{as} \quad x \to +\infty.$$

Therefore,

$$\frac{\Phi'(\sigma)}{\Phi(\sigma)} \uparrow +\infty, \quad \frac{\Phi'(\sigma)}{\Phi^2(\sigma)} \downarrow 0 \quad \text{as} \quad \sigma \uparrow 0$$

and

$$\Phi(\varphi(x))\Phi'(\Phi^{-1}(x)) = (1 + o(1))x^2 \quad \text{as} \quad x \to +\infty.$$

Finally,

$$\frac{\Phi''(\sigma)\Phi(\sigma)}{(\Phi'(\sigma))^2} = 1 + \frac{2|\sigma|}{\varrho}.$$

Thus, the function

$$\Phi(\sigma) = \exp\left\{\frac{\varrho}{|\sigma|}\right\}$$

satisfies all conditions of Theorems 1 and 2. Since $\Psi(\sigma) = -|\sigma|(1 + |\sigma|/\varrho)$ we have

$$\begin{aligned}\Phi(\Psi(\sigma)) &= \exp\left\{\frac{\varrho}{|\sigma|}\left(1 + \frac{|\sigma|}{\varrho}\right)^{-1}\right\} \\ &= \exp\left\{\frac{\varrho}{|\sigma|}\left(1 - \frac{(1 + o(1))|\sigma|}{\varrho}\right)\right\} = \frac{(1 + o(1))}{e}\Phi(\sigma)\end{aligned}$$

as $\sigma \uparrow 0$ and, thus,

$$\Phi(\Psi(\varphi(x))) = \frac{(1+o(1))x\varrho}{e\ln^2 x}$$

as $x \to +\infty$, which implies equivalence of (5) and (20). Finally, since

$$\Phi'(\Phi^{-1}(x)) = \frac{x\ln^2 x}{\varrho}$$

the conditions (8) and $\ln n(t) = O(t\ln^{-2} t)$ $(t \to +\infty)$ are also equivalent. Corollary 1 is proved.

Corollary 2. *Let $p > 1$. For the relations*

$$\int\limits_{\sigma_0}^{+\infty} \frac{\sigma^{p-1}\ln M(\sigma, F)}{\exp\{\sigma^p\}} d\sigma < +\infty$$

and

$$\int\limits_{\sigma_0}^{+\infty} \frac{\sigma^{p-1}\ln \mu(\sigma, F)}{\exp\{\sigma^p\}} d\sigma < +\infty$$

to be equivalent for any function $f \in S(\Lambda, +\infty)$, it is sufficient that

$$\int\limits_{t_0}^{\infty} \frac{\ln^{(p-1)/p} t}{t^2} \ln n(t) dt < +\infty \tag{21}$$

and necessary that $\ln n(t) = O(t\ln^{-(p-1)/p} t)$ $(t \to +\infty)$.

Indeed, for

$$\Phi(\sigma) = \exp\{\sigma^p\} \qquad (\sigma \geq \sigma_0)$$

we have

$$\Phi'(\sigma) = p\sigma^{p-1}\exp\{\sigma^p\}, \quad \Phi^{-1}(x) = (\ln x)^{1/p}, \quad \Phi'(\Phi^{-1}(x)) = px(\ln x)^{(p-1)/p}$$

and since

$$p(\varphi(x))^{p-1}\exp\{(\varphi(x))^p\} \equiv x$$

we also have

$$\varphi(x) = (1+o(1))(\ln x)^{1/p}$$

and

$$\Phi(\varphi(x)) = (x/p)(\varphi(x))^{-(p-1)} = (1+o(1))(x/p)(\ln x)^{-(p-1)/p} \quad \text{as} \quad x \to +\infty.$$

Therefore,

$$\frac{\Phi'(\sigma)}{\Phi(\sigma)} \uparrow +\infty, \quad \frac{\Phi'(\sigma)}{\Phi^2(\sigma)} \downarrow 0 \quad \text{as} \quad \sigma \to +\infty$$

and

$$\Phi(\varphi(x))\Phi'(\Phi^{-1}(x)) = (1+o(1))x^2 \quad \text{as} \quad x \to +\infty.$$

Finally,

$$\frac{\Phi''(\sigma)\Phi(\sigma)}{(\Phi'(\sigma))^2} = 1 + \frac{(p-1)}{p\sigma^p}.$$

Thus, the function $\Phi(\sigma) = \exp\{\sigma^p\}$ satisfies all conditions of Theorems 1 and 2. Since $\Psi(\sigma) = \sigma - 1/(p\sigma^{p-1})$ we have

$$\Phi(\Psi(\sigma)) = \exp\left\{\sigma^p\left(1 - \frac{1}{p\sigma^p}\right)^p\right\} = \exp\left\{\sigma^p\left(1 - \frac{1+o(1)}{\sigma^p}\right)\right\} = \frac{(1+o(1))}{e}\Phi(\sigma)$$

as $\sigma \to +\infty$ and, thus,

$$\Phi(\Psi(\varphi(x))) = \frac{1+o(1)}{e}\frac{x}{p}(\ln x)^{-(p-1)/p} \quad \text{as} \quad x \to +\infty,$$

whence it follows that (5) and (21) are equivalent. Finally, since $\Phi'(\Phi^{-1}(x)) = px(\ln x)^{(p-1)/p}$ the conditions (8) and $\ln n(t) = O(t\ln^{-(p-1)/p} t)$ $(t \to +\infty)$ are also equivalent. Corollary 2 is proved.

We remark that (21) holds provided $\ln n(t) = O(t\ln^{-\alpha} t)$ $(t \to +\infty)$ with $\alpha > (2p-1)/p$.

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ZALEŻNOŚĆ MIĘDZY NAJWIĘKSZYM MODUŁEM I NAJWIĘKSZYM WYRAZEM SZEREGU DIRICHLETA W TERMINACH KLASY ZBIEŻNOŚCI

S t r e s z c z e n i e

Zbadana została zależność między wzrostem największego modułu i wzrostem największego wyrazu szeregu Dirichleta w terminach klasy zbieżności.