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NONLINEAR D'ALEMBERT EQUATION IN THE PSEUDO-EUCLIDEAN SPACE $R_{2,n}$ AND ITS SOLUTIONS

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We investigate the nonlinear D'Alembert equation in the pseudo-Euclidean space $R_{2,n}$ and construct new exact solutions containing arbitrary functions.

1. The nonlinear D'Alembert equation in the pseudo-Euclidean space $R_{2,n}$ has the form

$$\Box u + F(u) = 0, \tag{1}$$

where

$$\Box u = u_{11} + u_{22} - u_{33} - \ldots - u_{n+2,n+2}, \qquad u_{\mu\nu} = \frac{\partial^2 u}{\partial x_\mu} \partial x_\nu,$$

$$u = u(x), \qquad x = (x_1, x_2, \ldots, x_{n+2}), \qquad \mu, \nu = 1, 2, \ldots, n+2,$$

and $F(u)$ is a smooth function. Equation (1) is invariant with respect to the Poincaré algebra $AP(2, n)$ with basis elements

$$P_\alpha = \partial_\alpha, \qquad J_{\alpha\beta} = g^{\alpha\nu} x_\nu \partial_\beta - g^{\beta\nu} x_\nu \partial_\alpha, \tag{2}$$

where

$$\partial_\alpha = \frac{\partial}{\partial x_\alpha}, \qquad g_{11} = g_{22} = -g_{33} = \ldots = -g_{n+2,n+2}, \qquad \alpha, \beta, \nu = 1, 2, \ldots, n+2.$$

Exact solutions of Eq. (1) under various restrictions on the function $F(u)$ were constructed in [1–5]. In the present paper, we construct new broad classes of exact solutions of Eq. (1) containing arbitrary functions. To find them, we use the method proposed in [6, 7]. Its idea can be briefly formulated as follows: Consider a symmetric ansatz for Eq. (1) and assume that it has the form

$$u = f(x)\varphi(\omega_1, \ldots, \omega_k) + g(x), \tag{3}$$

where $\omega_1 = \omega_1(x_1, \ldots, x_{n+2}), \ldots, \omega_k = \omega_k(x_1, \ldots, x_{n+2})$ are new independent variables. Ansatz (3) selects a certain subset S of solutions of Eq. (1). We construct a new ansatz

$$u = f(x)\varphi(\omega_1, \ldots, \omega_k, \omega_{k+1}, \ldots, \omega_l) + g(x), \tag{4}$$

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which is a generalization of ansatz (3). New variables $\omega_{k+1}, \ldots, \omega_l$ in Eq. (4) are to be determined. We find them from the condition that the reduced equation corresponding to ansatz (3) coincides with the reduced equation corresponding to ansatz (4). Ansatz (4) selects a subset S_1 of solutions of Eq. (1) containing the subset S. If the solutions of the subset S are known, then it is also possible to construct the solutions of the subset S_1. The latter can be derived in the following way: Let $u = u(x, C_1, \ldots, C_t)$ be a multiparameter family of solutions of Eq. (1) having the form (3). We can obtain a more general family of solutions of Eq. (1) if the constants C_i in the solution $u = u(x, C_1, \ldots, C_t)$ are assumed to be arbitrary smooth functions of $\omega_{k+1}, \ldots, \omega_l$.

2. We select certain ansatzes of the type (3) and their generalizations of the type (4) for Eq. (1).

1°. The ansatz $u = \varphi(x_1, x_3, \omega)$, where $\omega = \omega(x_2, x_4, \ldots, x_{n+2})$ is an arbitrary solution of the system

$$\Box\omega = \frac{\partial^2\omega}{\partial x_2^2} - \frac{\partial^2\omega}{\partial x_4^2} - \ldots - \frac{\partial^2\omega}{\partial x_{n+2}^2} = 0,$$

$$(\nabla\omega)^2 = \left(\frac{\partial\omega}{\partial x_2}\right)^2 - \left(\frac{\partial\omega}{\partial x_4}\right)^2 - \ldots - \left(\frac{\partial\omega}{\partial x_{n+2}}\right)^2 = 0, \tag{5}$$

is a generalization of the symmetric ansatz $u = \varphi(x_1, x_3)$ and reduces Eq. (1) to the equation

$$\frac{\partial^2\varphi}{\partial x_1^2} - \frac{\partial^2\varphi}{\partial x_3^2} + F(\varphi) = 0. \tag{6}$$

A broad class of exact solutions of system (5) for $n \geq 3$, which are called Smirnov–Sobolev solutions, was constructed in [8, 9]. If $n = 2$, an arbitrary solution of system (5) has the form $f(x_2 \pm x_4)$, where f is a twice differentiable function. Note that system (5) was completely integrated in [10] for $n \geq 4$.

2°. The ansatz $u = \varphi(\omega_1, \omega_2, \omega_3)$, where $\omega_1 = x_1^2 + x_2^2 - x_3^2 - \ldots - x_k^2$, $\omega_2 = x_1 - x_k$, and $\omega_3 = (x_1 - x_k)/(x_2 - x_{k-1})$, $k \geq 4$, is a generalization of the symmetric ansatz $u = \varphi(\omega_1, \omega_2)$ and reduces Eq. (1) to the equation

$$4\omega_1\varphi_{11} + 4\omega_2\varphi_{12} + 2k\varphi_1 + F(\varphi) = 0,$$

where

$$\varphi_{\mu\nu} = \frac{\partial^2\varphi}{\partial\omega_\mu\partial\omega_\nu}, \quad \mu, \nu = 1, 2.$$

3°. The ansatz $u = \varphi(\omega_1, \omega_2, \omega_3, \omega_4)$, where $\omega_1 = x_1^2 + x_2^2 - x_3^2 - \ldots - x_k^2$, $\omega_2 = x_1 - x_k$, $\omega_3 = x_2 - x_{k-1}$, and $\omega_4 = (x_1 - x_k)/(x_2 - x_{k-1})$, $k \geq 4$, is a generalization of the symmetric ansatz $u = \varphi(\omega_1, \omega_2, \omega_3)$ and reduces Eq. (1) to the equation

$$4\omega_1\varphi_{11} + 4\omega_2\varphi_{12} + 4\omega_3\varphi_{13} + 2k\varphi_1 + F(\varphi) = 0.$$

4°. The ansatz $u = \varphi(x_k, \dots, x_{n+2}, \omega)$, $4 \le k \le n+2$, $n \ge 4$, where ω is an arbitrary solution of the system

$$\frac{\partial^2\omega}{\partial x_1^2} + \frac{\partial^2\omega}{\partial x_2^2} - \frac{\partial^2\omega}{\partial x_3^2} - \dots - \frac{\partial^2\omega}{\partial x_{k-1}^2} = 0,$$

$$\left(\frac{\partial\omega}{\partial x_1}\right)^2 + \left(\frac{\partial\omega}{\partial x_2}\right)^2 - \left(\frac{\partial\omega}{\partial x_3}\right)^2 - \dots - \left(\frac{\partial\omega}{\partial x_{k-1}}\right)^2 = 0 \tag{7}$$

is a generalization of the symmetric ansatz $u = \varphi(x_k, \dots, x_{n+2})$ and reduces Eq. (1) to the equation

$$-\frac{\partial^2\varphi}{\partial x_k^2} - \dots - \frac{\partial^2\varphi}{\partial x_{n+2}^2} + F(\varphi) = 0.$$

Note that, in the case $k = 4$, system (7) is a system of the type (5).

5°. The ansatz $u = \varphi(\omega_1, \omega_2, \dots, \omega_l, \omega)$, where $\omega_1 = x_1^2 + x_2^2 - x_3^2 - \dots - x_k^2$, $\omega_2 = x_{k+1}, \dots, \omega_l = x_{k+l-1}$, $k \ge 3$, $k + l - 1 \le n + 2$, and ω is an arbitrary solution of the system

$$\Box\omega = \frac{\partial^2\omega}{\partial x_1^2} + \frac{\partial^2\omega}{\partial x_2^2} - \frac{\partial^2\omega}{\partial x_3^2} - \dots - \frac{\partial^2\omega}{\partial x_k^2} = 0,$$

$$(\nabla\omega)^2 = \left(\frac{\partial\omega}{\partial x_1}\right)^2 + \left(\frac{\partial\omega}{\partial x_2}\right)^2 - \left(\frac{\partial\omega}{\partial x_3}\right)^2 - \dots - \left(\frac{\partial\omega}{\partial x_k}\right)^2 = 0,$$

$$\nabla\omega_1 \cdot \nabla\omega = x_1\frac{\partial\omega}{\partial x_1} + x_2\frac{\partial\omega}{\partial x_2} + \dots + x_k\frac{\partial\omega}{\partial x_k} = 0,$$

is a generalization of the symmetric ansatz $u = \varphi(\omega_1, \omega_2, \dots, \omega_l)$ and reduces Eq. (1) to the equation

$$4\omega_1\frac{\partial^2\varphi}{\partial\omega_1^2} + 2k\frac{\partial\varphi}{\partial\omega_2} - \frac{\partial^2\varphi}{\partial\omega_2^2} - \dots - \frac{\partial^2\varphi}{\partial\omega_l^2} + F(\varphi) = 0.$$

6°. The ansatz $u = \varphi(\omega_1, \omega_2, \omega_3)$, where $\omega_1 = x_1$, $\omega_2 = x_2^2 - x_3^2 - \dots - x_k^2$, and ω_3 is an arbitrary solution of the system

$$\frac{\partial^2\omega_3}{\partial x_2^2} - \frac{\partial^2\omega_3}{\partial x_3^2} - \dots - \frac{\partial^2\omega_3}{\partial x_k^2} = 0,$$

$$\left(\frac{\partial\omega_3}{\partial x_2}\right)^2 - \left(\frac{\partial\omega_3}{\partial x_3}\right)^2 - \dots - \left(\frac{\partial\omega_3}{\partial x_k}\right)^2 = 0, \tag{8}$$

is a generalization of the symmetric ansatz $u = \varphi(\omega_1, \omega_2)$ and reduces Eq. (1) to the equation

$$\frac{\partial^2\varphi}{\partial\omega_1^2} + 4\omega_2\frac{\partial^2}{\partial\omega_2^2} + 2(k-1)\frac{\partial\varphi}{\partial\omega_2} + F(\varphi) = 0.$$

7°. The ansatz $u = \varphi(\omega_1, \omega_2, \omega_3, \ldots, \omega_l, \omega)$, where $\omega_1 = x_1$, $\omega_2 = x_2^2 - x_3^2 - \ldots - x_k^2$, $\omega_3 = x_{k+1}, \ldots, \omega_l = x_{k+l-2}$, $k \geq 4$, $k + l - 2 \leq n + 2$, and $\omega = \omega(x_2, x_3, \ldots, x_k)$ is an arbitrary solution of the system

$$\Box\omega = 0, \quad (\nabla\omega)^2 = 0, \quad \nabla\omega_2 \cdot \nabla\omega = 0, \tag{9}$$

is a generalization of the symmetric ansatz $u = \varphi(\omega_1, \omega_2, \omega_3, \ldots, \omega_l)$ and reduces Eq. (1) to the equation

$$\frac{\partial^2\varphi}{\partial\omega_1^2} + 4\omega_2\frac{\partial^2\varphi}{\partial\omega_2^2} + 2(k-2)\frac{\partial\varphi}{\partial\omega_2} - \frac{\partial^2\varphi}{\partial\omega_3^2} - \ldots - \frac{\partial^2\varphi}{\partial\omega_l^2} + F(\varphi) = 0.$$

If $k = 4$, an arbitrary solution of Eq. (9) is an arbitrary smooth function $f(x)$, where

$$t = \frac{x_2x_3 \pm x_4\sqrt{x_3^2 + x_4^2 - x_2^2}}{x_3^2 + x_4^2}.$$

The presented ansatzes 1°–7° can be efficiently used for the construction of broad classes of exact solutions of Eq. (1) under various restrictions on the function $F(u)$.

3. Let us construct several classes of exact solutions of the Liouville equation

$$\Box u + \lambda \exp u = 0. \tag{10}$$

Equation (10) is invariant with respect to the extended Poincaré algebra $A\tilde{P}(2, n)$, the basis of which is formed by operators (2) and the operator

$$D = -x_1\partial_1 - x_2\partial_2 - \ldots - x_{n+2}\partial_{n+2} + 2\partial_u.$$

We use the following notation [3]:

$$A_1 = -J_{14} + J_{23}, \quad A_2 = \frac{1}{2}(J_{12} + J_{34} - J_{13} - J_{24}), \quad D_1 = J_{14} + J_{24},$$

$$A_3 = \frac{1}{2}(J_{12} + J_{34} + J_{13} + J_{24}), \quad T = J_{12} - J_{24} + J_{13} - J_{34}, \tag{11}$$

$$N_1 = P_1 + P_4, \quad N_2 = P_2 + P_3, \quad Y_1 = P_1 - P_4, \quad Y_2 = P_2 - P_3.$$

Let $y_1 = x_1 + x_4$, $y_2 = x_1 - x_4$, $y_3 = x_2 + x_3$, and $y_4 = x_2 - x_3$.

Operators (11) belong to the algebra $A\tilde{P}(2, 2)$, which is a subalgebra of the algebra $A\tilde{P}(2, n)$. For the classification of subalgebras of the algebra $A\tilde{P}(2, 2)$ according to their ranks, see [3]. In what follows, we use several subalgebras of rank 3 of the algebra $A\tilde{P}(2, 2)$ to construct exact solutions of Eq. (10).

1. The subalgebra $\langle A_1 + \alpha D_1, D + \beta D_1\rangle$, $\alpha \geq 0$, $\alpha \neq 1$. Its principal invariants are the functions

$$u + \ln\omega_1, \qquad \omega = \ln\omega_1^{\beta+1}\omega_2^{-(1+\alpha)},$$

where $\omega_1 = y_1 y_2 + y_3 y_4$ and $\omega_2 = y_2 y_4^{(1-\alpha)/(1+\alpha)}$. The ansatz $u = \varphi(\omega) - \ln\omega_1$ reduces Eq. (10) to the equation

$$4(\beta^2 - 1)\ddot{\varphi} + 4(\beta + 1)\dot{\varphi} + \lambda\exp\varphi - 4 = 0,$$

which, in the case $\beta = 1$, possesses a solution

$$\varphi = \ln\frac{4Ce^{\omega/2}}{1 + \lambda Ce^{\omega/2}}.$$

Therefore, the function

$$u = \ln\frac{4C\omega_2^{-(1+\alpha)/2}}{1 + \lambda C\omega_1\omega_2^{-(1+\alpha)/2}}$$

is a solution of Eq. (10). Hence, in view of 3°, we obtain the following family of exact solutions of the Liouville equation:

$$u = \ln\frac{4h(t)}{x_2 - x_3 + \lambda h(t)(x_1^2 + x_2^2 - x_3^2 - x_4^2)},$$

where $h(t)$ is an arbitrary twice-differentiable function of $t = (x_1 - x_4)/(x_2 - x_3)$.

2. The subalgebra $\langle A_3 + N_1, T, D + \alpha A_1 + (1 + 3\alpha)D_1\rangle$, $\alpha \neq -1/2$. Its principal invariants are the functions

$$u + \frac{1}{2\alpha + 1}\ln\omega_1, \qquad \omega = \frac{\omega_2^{2\alpha+1}}{\omega_1^{2(\alpha+1)}},$$

where $\omega_1 = y_4$ and $\omega_2 = 4y_2^2 + 4y_2(y_1 y_2 + y_3 y_4)$. The ansatz

$$u = \varphi(\omega) - \frac{1}{2\alpha + 1}\ln\omega_1$$

reduces Eq. (10) to the equation

$$16(2\alpha + 1)e^{-\omega/(2\alpha+1)}(2\alpha\ddot{\varphi} + \dot{\varphi}) + \lambda\exp\varphi = 0.$$

If $\alpha = 0$, we obtain the equation

$$16e^{-\omega}\dot{\varphi} + \lambda\exp\varphi = 0,$$

and the function

$$u = \ln \frac{16\omega_1}{\lambda\omega_2 + C\omega_1^2}$$

is its solution. Hence, in view of 3°, we obtain the following family of exact solutions of the Liouville equation:

$$u = \ln \frac{4(x_2 - x_3)}{\lambda[(x_1 - x_4)^2 + (x_2 - x_3)(x_1^2 + x_2^2 - x_3^2 - x_4^2)] + h(t)(x_2 - x_3)^2},$$

where $h(t)$ is an arbitrary twice-differentiable function of $t = (x_1 - x_4)/(x_2 - x_3)$.

3. The subalgebra $\langle A_1 + D_1 + N_1, T, D + D_1 + \beta N_1 \rangle$. The principal invariants of this subalgebra are the functions

$$u + \ln\omega_1, \qquad \omega = \omega_2 - \ln\omega_1^{\beta+1},$$

where

$$\omega_1 = y_2 \quad \text{and} \quad \omega_2 = \frac{y_1 y_2 + y_3 y_4}{y_2} + \ln y_4.$$

The ansatz $u = \varphi(\omega) - \ln\omega_1$ reduces the Liouville equation to the equation

$$-4\beta\ddot{\varphi} + 4\dot{\varphi} + \lambda\exp\varphi = 0,$$

which, in the case $\beta = 0$, has a solution

$$\varphi = -\ln\left(\frac{\lambda}{4}\omega + C\right).$$

Therefore, the function

$$u = -\ln\left\{\frac{\lambda}{4}\left[x_1^2 + x_2^2 - x_3^2 - x_4^2 + (x_1 - x_4)\ln\frac{x_2 - x_3}{x_1 - x_4}\right] + C(x_1 - x_4)\right\}$$

is a solution of Eq. (10). Hence, in view of 3°, we obtain the following family of exact solutions of the Liouville equation:

$$u = -\ln\left\{\frac{\lambda}{4}(x_1^2 + x_2^2 - x_3^2 - x_4^2) + (x_1 - x_4)h(t)\right\},$$

where $h(t)$ is an arbitrary twice-differentiable function of $t = (x_1 - x_4)/(x_2 - x_3)$.

4. The symmetric ansatz $u = \varphi(x_1, x_3)$ reduces Eq. (10) to the equation

$$\frac{\partial^2\varphi}{\partial x_1^2} - \frac{\partial^2\varphi}{\partial x_3} + \lambda\exp\varphi = 0. \tag{12}$$

The general solution of Eq. (12)

$$\varphi(x_1, x_3) = \ln\left\{-\frac{8}{\lambda}\frac{\dot{h}_1(x_1+x_3)\dot{h}_2(x_1-x_3)}{[h_1(x_1+x_3)+h_2(x_1-x_3)]^2}\right\},$$

where h_1 and h_2 are arbitrary differentiable functions, $\dot{h}_1$ and $\dot{h}_2$ are their derivatives with respect to the corresponding argument, and $\lambda h_1 h_2 < 0$, was obtained by Liouville in 1853. Thus, according to 1°, the Liouville equation possesses the following family of exact solutions:

$$u(x_1, x_3) = \ln\left\{-\frac{8}{\lambda}\frac{\dot{f}(x_1+x_3, \omega)\dot{g}(x_1-x_3, \omega)}{[f(x_1+x_3, \omega)+g(x_1-x_3, \omega)]^2}\right\}.$$

Here, $f(x_1+x_3, \omega)$ and $g(x_1-x_3, \omega)$ are arbitrary differentiable functions of two arguments x_1+x_3, ω and x_1-x_3, ω, respectively, $\dot{f}$ and $\dot{g}$ are their derivatives with respect to the first argument, and ω is an arbitrary solution of system (5).

5. The symmetric ansatz $u = \varphi(x_{n+2})$ reduces Eq. (10) to the equation

$$-\frac{d^2\varphi}{dx_{n+2}^2} + \lambda\exp\varphi = 0. \tag{13}$$

Integrating Eq. (13) and using 4°, we obtain the following families of exact solutions of the Liouville equation:

$$u = \ln\left\{\left(\frac{h_1(\omega)}{2\lambda}\sec^2\left[\frac{\sqrt{-h_1(\omega)}}{2}(x_{n+2}+h_2(\omega))\right]\right)\right\}, \qquad h_1(\omega) < 0, \quad \lambda < 0,$$

$$u = \ln\left\{-\frac{2h_1(\omega)h_2(\omega)\exp\sqrt{h_1(\omega)}\,x_{n+2}}{\lambda[1-h_2(\omega)\exp\sqrt{h_1(\omega)}\,x_{n+2}]^2}\right\}, \qquad h_1(\omega) > 0, \quad \lambda h_2(\omega) > 0,$$

$$u = -\ln\left\{\sqrt{\frac{\lambda}{2}}\,x_{n+2} + h(\omega)\right\},$$

where $h_1(\omega)$, $h_2(\omega)$, and $h(\omega)$ are arbitrary twice-differentiable functions and ω is an arbitrary solution of system (7) for $k = n+2$.

4. We now construct several classes of exact solutions of the nonlinear D'Alembert equation

$$\Box u + \lambda u^k = 0, \qquad k \neq 1. \tag{14}$$

Equation (14) is invariant with respect to the extended Poincaré algebra $A\tilde{P}(2, n)$, the basis of which is formed by operators (2) and the operator

$$D = -x_1\partial_1 - x_2\partial_2 - \ldots - x_{n+2}\partial_{n+2} + \frac{2u}{k-1}\partial_u.$$

We use several subalgebras of rank 3 of the algebra $A\tilde{P}(2, 2)$ to construct solutions of Eq. (14). We preserve notation (11) for the operators of the algebra $A\tilde{P}(2, 2)$.

By analogy with Sec. 3, using the subalgebras $\langle A_1 + D_1 + N_1, T, D + D_1 + \beta N_1\rangle$, $\langle A_3 + N_1, T, D + \alpha A_1 + (1 + 3\alpha)D_1\rangle$ $(\alpha \neq -1/2)$, $\langle A_3 + N_1, T, 2D - A_1 - D_1\rangle$, $\langle D_1, T, D + \alpha(A_2 + A_3)\rangle$ $(\alpha > 0)$, $\langle A_1 + \alpha D_1, T, D + \beta D_1\rangle$ $(\alpha \geq 0,\ \alpha \neq 1)$, and $\langle A_1 - D_1, A_3, T, D + \beta D_1\rangle$, we obtain the solutions

$$u = \frac{24\beta}{\lambda[x_1^2 + x_2^2 - x_3^2 - x_4^2 + (x_1 - x_4)(-\ln(x_1 - x_4)^{\beta} + h(t))]^2},$$

$$u^{1-k} = \frac{\lambda(k-1)^2}{16(k-2)}(x_2 - x_3)^{-1}\{[2(x_1 - x_4) + (x_1 + x_4)(x_2 - x_3)]^2$$

$$- (x_2 - x_3)^2[(x_1 + x_4)^2 - 4(x_2 + x_3)]\} + (x_2 - x_3)h(t),$$

$$u = \frac{16(x_2 - x_3)}{[h(t) - \lambda\ln(x_2 - x_3)][4(x_2 - x_4)^2 + 4(x_2 - x_3)(x_1^2 + x_2^2 - x_3^2 - x_4^2)]},$$

$$u^2 = \frac{8h(t)}{\lambda[1 - (x_1^2 + x_2^2 - x_3^2 - x_4^2)h(t)]^2},$$

$$u^{1-k} = \frac{\lambda(k-1)^2}{4(k-2)}(x_1^2 + x_2^2 - x_3^2 - x_4^2) + (x_2 - x_3)h(t),$$

$$u^{1-k} = (x_1^2 + x_2^2 - x_3^2 - x_4^2)\left\{\frac{\lambda(k-1)^2}{4(k-2)} + h(t)(x_2 - x_3)^{k-2}\right\},$$

respectively. Here, $h(t)$ is an arbitrary twice-differentiable function of $t = (x_1 - x_4)/(x_2 - x_3)$.

5. The solutions of the nonlinear D'Alembert and Liouville equations presented above can easily be generalized to the case of arbitrary n. Consider several examples.

The symmetric ansatz for the D'Alembert equation in the case $n = 3$ takes the form

$$u = (x_1^2 + x_2^2 - x_3^2 - x_4^2 - x_5^2)^{1/(1-k)}\varphi(\omega),$$

where $\omega = \ln\{(x_1^2 + x_2^2 - x_3^2 - x_4^2 - x_5^2)^{(\beta-1)/2}(x_1 - x_5)\}$.

This ansatz reduces Eq. (14) to the equation

$$(\beta - 1)(\beta - 3)\ddot{\varphi} + \left[-\frac{4(\beta-2)}{k-1} + 3(\beta - 1)\right]\dot{\varphi} - \frac{2(3k-5)}{(k-1)}\varphi + \lambda\varphi^k = 0,$$

which, in the cases $\beta = 1$ and $\beta = 3$, has the solutions

$$\varphi^{1-k} = \frac{\lambda(k-1)^2}{2(3k-5)} + C_1 e^{-\omega(3k-5)/2} \quad \text{and} \quad \varphi^{1-k} = \frac{\lambda(k-1)^2}{2(3k-5)} + C_1 e^{-\omega},$$

respectively. Hence, we obtain the following solutions of the D'Alembert equation:

$$u^{1-k} = \left\{\frac{\lambda(k-1)^2}{2(3k-5)} + C_1(x_1 - x_5)^{-(3k-5)/2}\right\}(x_1^2 + x_2^2 - x_3^2 - x_4^2 - x_5^2),$$

$$u^{1-k} = \frac{\lambda(k-1)^2}{2(3k-5)}(x_1^2 + x_2^2 - x_3^2 - x_4^2 - x_5^2) + \frac{C_1}{x_1 - x_5}. \tag{15}$$

Acting by the group transformation $\exp(\mathrm{ad}\, C_2 P_3)$ on these solutions, we derive the following solutions of Eq. (14):

$$u^{1-k} = \left\{\frac{\lambda(k-1)^2}{2(3k-5)} + C_1(x_1 - x_5)^{-(3k-5)/2}\right\}[x_1^2 + x_2^2 - (x_3 + C_2)^2 - x_4^2 - x_5^2],$$

$$u^{1-k} = \frac{\lambda(k-1)^2}{2(3k-5)}[x_1^2 + x_2^2 - (x_3 + C_2)^2 - x_4^2 - x_5^2] + \frac{C_1}{x_1 - x_5}.$$

Hence, in view of 3°, we obtain the following families of exact solutions of the D'Alembert equation:

$$u^{1-k} = \left\{\frac{\lambda(k-1)^2}{2(3k-5)} + h_1(t)(x_1 - x_5)^{-(3k-5)/2}\right\}[x_1^2 + x_2^2 - (x_3 + h_2(t))^2 - x_4^2 - x_5^2],$$

$$u^{1-k} = \frac{\lambda(k-1)^2}{2(3k-5)}[x_1^2 + x_2^2 - (x_3 + h_2(t))^2 - x_4^2 - x_5^2] + \frac{h_1(t)}{x_1 - x_5},$$

where $h_1(t)$ and $h_2(t)$ are arbitrary twice-differentiable functions of $(x_1 - x_5)/(x_2 - x_4)$.

Acting by the group transformation $\exp[\mathrm{ad}(C_2 P_1 + C_3 P_5)]$ on solutions (15), we obtain the following solutions of Eq. (14):

$$u^{1-k} = \left\{\frac{\lambda(k-1)^2}{2(3k-5)} + C_1(x_1 - x_5 + C_2 - C_3)^{-(3k-5)/2}\right\}[(x_1 + C_2)^2 + x_2^2 - x_3^2 - x_4^2 - (x_5 + C_3)^2],$$

$$u^{1-k} = \frac{\lambda(k-1)^2}{2(3k-5)}[(x_1 + C_2)^2 + x_2^2 - x_3^2 - x_4^2 - (x_5 + C_3)^2] + \frac{C_1}{x_1 - x_5 + C_2 - C_3}.$$

Thus, on the basis of 7°, we obtain the following families of exact solutions of the D'Alembert equation:

$$u^{1-k} = \left\{\frac{\lambda(k-1)^2}{2(3k-5)} + h_1(t)(x_1 - x_5 + h_2(t) - h_3(t))^{-(3k-5)/2}\right\}[(x_1 + h_2(t))^2 + x_2^2 - x_3^2 - x_4^2 - (x_5 + h_3(t))^2],$$

$$u^{1-k} = \frac{\lambda(k-1)^2}{2(3k-5)}[(x_1 + h_2(t))^2 + x_2^2 - x_3^2 - x_4^2 - (x_5 + h_3(t))^2]\ \frac{h_1(t)}{x_1 - x_5 + h_2(t) - h_3(t)},$$

where $h_1(t)$, $h_2(t)$, and $h_3(t)$ are arbitrary twice-differentiable functions of

$$t = \frac{x_2 x_3 \pm x_4\sqrt{x_3^2 + x_4^2 - x_2^2}}{x_3^2 + x_4^2}.$$

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