

# Separation of variables for nonlinear equations of hyperbolic and Korteweg de Vries type

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## Abstract

We propose substitutions that have been used for construction of wide classes of exact solutions with the generalized separation of variables for the nonlinear equations of hyperbolic and Korteweg de Vries type (KdV-type). These solutions cannot be obtained by means of S. Lie method or by the method of conditional symmetries.

*Key words:* nonlinear hyperbolic equations; Korteweg de Vries equations; generalized separation of variables; Lie method

## 1 Introduction

One of effective method for construction of exact solutions of linear PDE is separation of variables. For equations with two independent variables  $x$ ,  $t$  and a dependent variable  $u$ , this method is based on finding of exact solutions in the form of product of functions of different arguments

$$u = a(x)b(t). \quad (1)$$

In the paper [1] the method is described for construction of exact solutions of nonlinear PDE which generalizes the classic variable separation method. The solutions are sought in the form of a finite sum of  $k$  terms:

$$u(x, t) = \sum_{i=1}^k f_i(t)a_i(x), \quad (2)$$

where  $f_i(t)$ ,  $a_i(x)$  are smooth functions that have to be determined. Exact solutions in the form of generalized separation of variables (2) that contain more than two terms are given in the papers [2–5]. The results given in [3–5] are based on finding of finite dimensional subspaces that are invariant under a corresponding nonlinear differential operator.

Substitution (1) can be considered as an ansatz which reduces the equation under study to an ordinary differential equation with unknown function  $a = a(x)$  (or with unknown function  $b = b(t)$ ). Using the idea of reduction, in the present paper we propose the following generalization of substitution (1) for nonlinear equations:

$$u = \sum_{i=1}^m \omega_i(t) a_i(x) + f(x, t), \quad m \geq 1. \quad (3)$$

Substitution (3) contains unknown function  $f(x, t)$ ,  $m$  unknown functions  $a_i(x)$  and  $m$  unknown functions  $\omega_i(t)$ . They are found from the condition that substitution (3) reduces the equation under consideration to a system of  $m$  ordinary differential equations with unknown functions  $\omega_i(t)$ . If  $m = 1$  in substitution (3) then this system is reduced to an ordinary differential equation with unknown function  $\omega_1(t)$ . In many cases substitution (3) simplifies finding of exact solutions in the form of generalized separation of variables (2).

If in a given differential equation we perform the substitution  $x = x(u, t)$ , then we have a differential equation with the independent variables  $u, t$  and the dependent variable  $x$ . For construction solutions of such equation, we can use the substitution

$$x = \sum_{i=1}^m \omega_i(t) a_i(u) + f(u, t), \quad (4)$$

which is obtained from substitution (3) as a result of performing of replacement  $u \mapsto x$ ,  $x \mapsto u$ ,  $t \mapsto t$ .

In this paper we use substitutions of type (3), (4) for construction of exact solutions of the nonlinear hyperbolic type equation

$$\frac{\partial^2 u}{\partial t^2} = au \frac{\partial^2 u}{\partial x^2} + b \left( \frac{\partial u}{\partial x} \right)^2 + c \quad (5)$$

and of KdV-type equation

$$\frac{\partial u}{\partial t} + F(u) \left( \frac{\partial u}{\partial x} \right)^k + \frac{\partial^3 u}{\partial x^3} = 0, \quad (6)$$

where  $k$  is a real parameter. Equation (5) was the subject of investigation in the papers [4,5], and equation (6) was investigated in the paper [6] using the conditional symmetry method. Solutions of equations (5) and (6) given in the present work can not be obtained by means of the group analysis method.

## 2 Separation of variables for equation (5)

For construction of exact solutions of equation (5), we use the substitution

$$u = \omega(t) + f(x, t), \quad (7)$$

which is the particular case of general substitution (3). This substitution is a generalization of the following one

$$u(x, t) = \varphi(x) + \psi(t)$$

that is often used for finding of exact solutions of nonlinear PDE. After substitution (7) to (5), we get the equation

$$\omega'' + f_{tt} = a(\omega + f)f_{xx} + bf_x^2 + c, \quad (8)$$

which has to be an ordinary differential equation with the unknown function  $\omega = \omega(t)$ . It follows that the coefficient  $af_{xx}$  at  $\omega$  in equation (8) can be written in the form of  $af_{xx} = 2a\mu_2(t)$ , and then

$$f = \mu_2(t)x^2 + \mu_1(t)x + \tilde{\mu}_0(t),$$

where  $\tilde{\mu}_0(t)$ ,  $\mu_1(t)$ ,  $\mu_2(t)$  are some functions of  $t$ . From (7) we obtain the substitution

$$u = \mu_2(t)x^2 + \mu_1(t)x + \mu_0(t), \quad (9)$$

$\mu_0(t) = \tilde{\mu}_0(t) + \omega(t)$ . This substitution was found in [4] on the base of analysis of structure of invariant subspaces for the operator which corresponds to right side of equation (5).

Let us consider the particular case of equation (5):

$$\frac{\partial^2 u}{\partial t^2} = au \frac{\partial^2 u}{\partial x^2} + b \left( \frac{\partial u}{\partial x} \right)^2. \quad (10)$$

For construction of exact solutions of equation (10) we use the substitution:

$$u = \omega(t)d(x) + f(x, t), \quad (11)$$

which is a particular case of the general substitution (3). Substitution (11) reduces (10) to the equation

$$\omega''d - \omega(af_{xx}d + 2bf_xd' + af d'') - \omega^2(add'' + bd'^2) + f_{tt} - aff_{xx} - bf_x^2 = 0. \quad (12)$$

Equation (12) has to be an ordinary differential equation with unknown function  $\omega = \omega(t)$ . It follows that

$$add'' + bd'^2 = \beta d, \quad \beta \in \mathbb{R}, \quad (13)$$

$$adf_{xx} + 2bd'f_x + ad''f = \tilde{\gamma}(t)d. \quad (14)$$

As a result, the problem of finding of exact solutions in the form of (11) for equation (10) is reduced to integration of the system of two ordinary differential equations, one of which is linear. Equation (13) has the following particular solution:

$$d = x^\alpha, \quad a = \frac{\alpha}{1-\alpha}b, \quad \beta = 0, \quad \alpha \neq 1. \quad (15)$$

Substituting  $d = x^\alpha$  into equation (14), we have

$$x^2f_{xx} + 2(1-\alpha)xf_x + \alpha(\alpha-1)f = \gamma(t)x^2, \quad (16)$$

where  $\gamma(t) = \frac{1-\alpha}{b}\tilde{\gamma}(t)$ .

Let us consider three cases.

a) Case  $\alpha = 2$ . From (15), it follows that  $a = -2b$ . The general solution of equation (16) for  $\alpha = 2$  is the function

$$f = \gamma(t)x^2 \ln|x| + \tilde{\beta}(t)x^2 + \delta(t)x,$$

and therefore, according to (11), we have

$$u = \gamma(t)x^2 \ln|x| + \beta(t)x^2 + \delta(t)x, \quad (17)$$

where  $\beta(t) = \tilde{\beta}(t) + \omega(t)$ . Substituting (17) to (10), we find the equality  $\delta(t) = 0$  and the system of the equations for determining of the functions  $\beta(t)$  and  $\gamma(t)$ :

$$\gamma'' = -2b\gamma^2, \quad \beta'' = -2b\beta\gamma + b\gamma^2. \quad (18)$$

System (18) has the following particular solution:

$$\gamma = -\frac{3}{b}t^{-2}, \quad \beta = c_1t^3 + c_2t^{-2} - \frac{9}{5b}t^{-2} \ln |t|.$$

As a result, we find such exact solution of equation (10) for  $a = -2b$ :

$$u = -\frac{3}{b}t^{-2}x^2 \ln |x| + \left( c_1t^3 + c_2t^{-2} - \frac{9}{5b}t^{-2} \ln |t| \right) x^2,$$

where  $c_1, c_2$  are arbitrary constants.

b) Case  $\alpha = 3$ . From (15), it follows that  $a = -\frac{3}{2}b$ . The general solution of equation (16) for  $\alpha = 3$  is the function

$$f = -\gamma(t)x^2 \ln |x| + \beta(t)x^2 + \tilde{\delta}(t)x^3,$$

and therefore according to (11) we have

$$u = -\gamma(t)x^2 \ln |x| + \beta(t)x^2 + \delta(t)x^3, \quad (19)$$

where  $\delta(t) = \tilde{\delta}(t) + \omega(t)$ . Substituting (19) to equation (10) we find the equality  $\gamma(t) = 0$ , and then substitution (19) has the form

$$u = \beta(t)x^2 + \delta(t)x^3.$$

The substitution  $u = \beta(t)x^2 + \delta(t)x^3$  is a particular case of the substitution having a more general form

$$u = \mu_3(t)x^3 + \mu_2(t)x^2 + \mu_1(t)x + \mu_0(t),$$

which is given in [5].

c) Case  $\alpha \neq 1; 2; 3$ . The general solution of equation (16) is the function

$$f = \frac{\gamma(t)}{(\alpha - 2)(\alpha - 3)}x^2 + \tilde{\mu}_2(t)x^\alpha + \mu_0(t)x^{-1+\alpha},$$

and therefore according to (11) we have

$$u = \mu_1(t)x^2 + \mu_2(t)x^\alpha + \mu_0(t)x^{-1+\alpha}, \quad (20)$$

where the unknown functions  $\mu_0(t), \mu_1(t) = \frac{\gamma(t)}{(\alpha - 2)(\alpha - 3)}$ ,  $\mu_2(t) = \tilde{\mu}_2(t) + \omega(t)$  have to be determined. Substituting (20) into equation

(10), we find the equality  $\mu_0(t) = 0$  and the system of the equations to determine of the functions  $\mu_1(t)$  and  $\mu_2(t)$ :

$$\mu_1'' = (2a + 4b)\mu_1^2, \quad (21)$$

$$\mu_2'' = \left( -\frac{a^2b}{(a+b)^2} + \frac{2a^2 + 6ab}{a+b} \right) \mu_1\mu_2. \quad (22)$$

A particular solution of equation (21) is the function:

$$\mu_1 = \frac{3}{a+2b}t^{-2}. \quad (23)$$

Substituting (23) to (22) and using relations (15), we obtain the linear equation to determine the function  $\mu_2(t)$ :

$$t^2\mu_2'' = (9\alpha - 3\alpha^2)\mu_2. \quad (24)$$

Equation (24) has the following solutions:

$$\mu_2 = t^{1/2}[c_1 \cos(\sigma \ln t) + c_2 \sin(\sigma \ln t)], \text{ if } \sigma^2 = 3\alpha^2 - 9\alpha - \frac{1}{4} > 0,$$

$$\mu_2 = t^{1/2}[c_1 t^\sigma + c_2 t^{-\sigma}], \text{ if } \sigma^2 = \frac{1}{4} - 3\alpha^2 + 9\alpha > 0,$$

$$\mu_2 = t^{1/2}[c_1 + c_2 \ln t], \text{ if } \alpha = \frac{9 \pm 2\sqrt{21}}{6},$$

where  $c_1, c_2$  are arbitrary constants.

So, equation (10) has the following exact solutions:

$$u = \frac{3}{a+2b}t^{-2}x^2 + t^{1/2}x^\alpha[c_1 \cos(\sigma \ln t) + c_2 \sin(\sigma \ln t)],$$

$$\text{if } \sigma^2 = 3\alpha^2 - 9\alpha - \frac{1}{4} > 0;$$

$$u = \frac{3}{a+2b}t^{-2}x^2 + t^{1/2}x^\alpha[c_1 t^\sigma + c_2 t^{-\sigma}],$$

$$\text{if } \sigma^2 = \frac{1}{4} - 3\alpha^2 + 9\alpha > 0;$$

$$u = \frac{3}{a+2b}t^{-2}x^2 + t^{1/2}x^\alpha[c_1 + c_2 \ln t],$$

if  $\alpha = \frac{9 \pm 2\sqrt{21}}{6}$ , where  $c_1, c_2$  are arbitrary constants.

Note that the substitution  $u = \mu_1 x^2 + \mu_2 x^4$  is a particular case of the substitution having more general form

$$u = \mu_4(t)x^4 + \mu_3(t)x^3 + \mu_2(t)x^2 + \mu_1(t)x + \mu_0(t),$$

which was found in [5].

The results obtained for equation (10) can be generalized to the equation

$$\frac{\partial^2 u}{\partial t^2} = au \frac{\partial^2 u}{\partial x^2} + b \left( \frac{\partial u}{\partial x} \right)^2 + \phi(t)u. \quad (25)$$

We have the following cases.

a) Case  $a = -2b$ . The substitution

$$u = \gamma(t)x^2 \ln |x| + \beta(t)x^2,$$

reduces equation (25) to the system

$$\gamma'' = -2b\gamma^2 + \phi\gamma, \quad \beta'' = -2b\beta\gamma + b\gamma^2 + \phi\beta.$$

b) Case  $a = -\frac{3}{2}b$ . The substitution

$$u = \beta(t)x^2 + \delta(t)x^3,$$

reduces equation (25) to the system

$$\delta'' = \phi\delta, \quad \beta'' = b\beta^2 + \phi\beta.$$

c) Case  $a \neq -b; -2b; -\frac{3}{2}b$ . The substitution

$$u = \mu_1(t)x^2 + \mu_2(t)x^\alpha,$$

where  $\alpha = \frac{a}{a+b}$  reduces equation (25) to the system

$$\begin{aligned} \mu_1'' &= (2a + 4b)\mu_1^2 + \phi\mu_1, \\ \mu_2'' &= \left( -\frac{a^2b}{(a+b)^2} + \frac{2a^2 + 6ab}{a+b} \right) \mu_1\mu_2 + \phi\mu_2. \end{aligned}$$

### 3 Separation of variables for equation (6)

Let us find out functions  $F(u)$  for which equation (6) admits the substitution in the form

$$x = \omega_1(t)d(u) + \omega_2(t). \quad (26)$$

Substitution (26) is a particular case of general one (4). We determine the functions  $\omega_1(t)$ ,  $\omega_2(t)$  and  $d(u)$  from the condition that substitution (26) reduces the equation (6) to a system of two ordinary differential equations with the unknown functions  $\omega_1 = \omega_1(t)$ ,  $\omega_2 = \omega_2(t)$ . Let us use substitution (26) into equation (6):

$$-\frac{\omega'_1}{\omega_1} \frac{d}{d'} - \frac{\omega'_2}{\omega_1} \frac{1}{d'} + \frac{1}{\omega_1^k} \frac{F(u)}{(d')^k} - \frac{1}{\omega_1^3} \frac{d'''}{(d')^4} + \frac{1}{\omega_1^3} \frac{3(d'')^2}{(d')^5} = 0. \quad (27)$$

The functions  $\frac{d}{d'}$ ,  $\frac{1}{d'}$  at  $-\frac{\omega'_1}{\omega_1}$ ,  $-\frac{\omega'_2}{\omega_1}$  in equation (27) are linearly independent. Let  $k \neq 3$  in equation (27). We lay down the conditions on the coefficients at the functions  $\frac{1}{\omega_1^k}$ ,  $\frac{1}{\omega_1^3}$  so that they can be represented as a linear combination of the functions  $\frac{d}{d'}$ ,  $\frac{1}{d'}$  on the field of real numbers. So we have

$$-\frac{d'''}{(d')^4} + 3\frac{(d'')^2}{(d')^5} = \lambda \frac{d}{d'} + \mu \frac{1}{d'}, \quad \lambda, \mu \in \mathbb{R},$$

or

$$-d'd''' + 3(d'')^2 = \lambda d(d')^4 + \mu(d')^4. \quad (28)$$

Taking into account (28), we find from equation (27) that

$$F(u) = [\omega'_1 \omega_1^{k-1} - \lambda \omega_1^{k-3}] d(d')^{k-1} + [\omega'_2 \omega_1^{k-1} - \mu \omega_1^{k-3}] (d')^{k-1},$$

and then

$$\omega'_1 \omega_1^{k-1} - \lambda \omega_1^{k-3} = \lambda_1, \quad \omega'_2 \omega_1^{k-1} - \mu \omega_1^{k-3} = \lambda_2,$$

where  $\lambda_1$ ,  $\lambda_2$  are constants. So, in the case  $k \neq 3$  equation (6) admits substitution (26) if the function  $F(u)$  has the form

$$F(u) = \lambda_1 d(d')^{k-1} + \lambda_2 (d')^{k-1}, \quad (29)$$

where  $d = d(u)$  is an arbitrary solution of equation (28). Let us consider the following particular solutions of equation (28):

- a)  $d = \ln u$ , if  $\lambda = 0$ ,  $\mu = 1$ ;
- b)  $d = u^{1/2}$ , if  $\lambda = \mu = 0$ ;
- c)  $d = \arcsin u$ , if  $\lambda = 0$ ,  $\mu = -1$ ;
- d)  $d = \operatorname{Arsh} u$ , if  $\lambda = 0$ ,  $\mu = 1$ .

In the case a) we get from (29) that

$$F(u) = (\lambda_1 \ln u + \lambda_2)u^{1-k}, \quad (30)$$

and the solutions of equation (6) are

$$u = \exp \left[ -\frac{k(k\lambda_1)^{-3/k}}{k-2} t^{(k-3)/k} + ct^{-1/k} + (k\lambda_1 t)^{-1/k} x - \frac{\lambda_2}{\lambda_1} \right], \quad k \neq 2,$$

$$u = \exp \left[ -(2\lambda_1)^{-3/2} t^{-1/2} \ln t + ct^{-1/2} + (2\lambda_1 t)^{-1/2} x - \frac{\lambda_2}{\lambda_1} \right], \quad k = 2,$$

where  $c$  is an arbitrary constant. The first of these functions is also a solution of equation (6) in the case  $k = 3$ , that follows from consideration of this case which is considered below.

Similarly, we determine that in the case b)

$$F(u) = \lambda'_1 u^{(2-k)/2} + \lambda'_2 u^{(1-k)/2}, \quad \lambda'_1 = 2^{1-k} \lambda_1, \quad \lambda'_2 = 2^{1-k} \lambda_2 \quad (31)$$

in the case c)

$$F(u) = (\lambda_1 \arcsin u + \lambda_2)(1 - u^2)^{(1-k)/2}, \quad (32)$$

and in the case d)

$$F(u) = (\lambda_1 \operatorname{Arsh} u + \lambda_2)(1 + u^2)^{(1-k)/2}. \quad (33)$$

Functions (30)–(33) and the solutions of equation (6) corresponding to them have been obtained by the method of conditional symmetry in [6].

Let  $k = 3$  in equation (6). We find from equation (27)

$$F(u) = \omega'_1 \omega_1^2 d(d')^2 + \omega'_2 \omega_1^2 (d')^2 + \frac{d'''}{d'} - \frac{3(d'')^2}{(d')^2},$$

and then as a consequence of linear independence of the functions  $d(d')^2$  and  $(d')^2$  we have

$$\omega'_1 \omega_1^2 = \lambda_1, \quad \omega'_2 \omega_1^2 = \lambda_2, \quad (34)$$

where  $\lambda_1, \lambda_2$  are constants. So, in the case  $k = 3$  equation (6) admits substitution (26) if the function  $F(u)$  has the form

$$F(u) = \lambda_1 d(d')^2 + \lambda_2 (d')^2 + \frac{d'''}{d'} - \frac{3(d'')^2}{(d')^2}, \quad (35)$$

where  $d = d(u)$  is an arbitrary smooth function. For example, if  $d = \exp u$  then from (35) we get

$$F(u) = \lambda_1 \exp(3u) + \lambda_2 \exp(2u) - 2,$$

and then owing to (26), (34) the equation

$$\frac{\partial u}{\partial t} + [\lambda_1 \exp(3u) + \lambda_2 \exp(2u) - 2] \left( \frac{\partial u}{\partial x} \right)^3 + \frac{\partial^3 u}{\partial x^3} = 0$$

has an exact solution

$$u = \ln \left[ \frac{x + c_1}{(3\lambda_1 t + c_2)^{1/3}} - \frac{\lambda_2}{\lambda_1} \right],$$

where  $c_1, c_2$  are arbitrary constants.

## 4 Conclusion

All exact solutions of equations (5) represented in section 2 of this paper are solutions on invariant subspaces in the meaning of book [5]. Substitution (3) and substitution obtained from it as a result of rearrangement of the variables  $u, x, t$  can be also used for finding of exact solutions of other nonlinear partial differential equations.

As a rule, substitution (3) can be applied to an equation with the polynomial nonlinearity. If finding of a solution of the form (3) is impossible, then, at first, we look for transformations that reduce an equation under consideration to an equation with polynomial nonlinearity, in particular with quadratic nonlinearity, and then we search solutions of the latter equation in form (3). In the papers [3, 4, 7] many nonlinear equations were described that are reduced to equations with quadratic nonlinearity by means of an appropriate transformation.

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