axioms

Article

On the Relative Φ-Growth of Hadamard Compositions of Dirichlet Series

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Abstract: For the Dirichlet series $F(s) = \sum\limits_{n=1}^{\infty} f_n \exp\{s\lambda_n\}$, which is the Hadamard composition of the genus m of similar Dirichlet series $F_j(s)$ with the same exponents, the growth with respect to the function $G(s)$ given as the Dirichlet series is studied in terms of the Φ-type (the upper limit of $M_G^{-1}(M_F(\sigma))/\Phi(\sigma)$ as $\sigma \uparrow A$) and convergence Φ-class defined by the condition $\int\limits_{\sigma_0}^{A} \frac{\Phi'(\sigma) M_G^{-1}(M_F(\sigma))}{\Phi^2(\sigma)} d\sigma < +\infty$, where $M_F(\sigma)$ is the maximum modulus of the function F at an imaginary line and A is the abscissa of the absolute convergence.

Keywords: Dirichlet series; Hadamard composition; Φ-type; convergence Φ-class

MSC: 30B50

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1. Introduction

Let f and g be entire transcendental functions and $M_f(r) = \max\{|f(z)| : |z| = r\}$. For the study of the comparative growth of the functions f and g, the mathematician Ch. Roy [1] used the relative order $\varrho_g[f] = \varlimsup\limits_{r\to+\infty} \dfrac{\ln M_g^{-1}(M_f(r))}{\ln r}$ and the lower relative order $\lambda_g[f] = \varliminf\limits_{r\to+\infty} \dfrac{\ln M_g^{-1}(M_f(r))}{\ln r}$ of the function f with respect to the function g; i.e., the growth of the function f with respect to the function g is identified with the growth of the function $M_g^{-1}(M_f(r))$ as $r \to +\infty$, where M_g^{-1} is the inverse function of M_g.

Research on the relative growth of entire functions was continued by S.K. Data, T. Biswas, and other mathematicians (see, for example, [2–5]) in terms of maximal terms, the Nevanlinna characteristic function, and the k-logarithmic orders. In particular, they [6] considered the relative growth of entire functions of two complex variables and examined [7] the relative growth of entire Dirichlet series by use of R-orders. Relative growth allows us to describe the properties of a very wide class of functions since we can freely choose the analytical function with respect to which we find growth characteristics. This provides a sufficiently flexible growth scale. The Hadamard composition is another notion intensively used in the paper. It is rich its unexpected connections and applications in the theory of functions. The Hadamard composition is very important in studying the properties of various classes of functions generated by power series and Dirichlet series. The notion is deeply connected with the convolution of functions. Many of its generalizations are known. Recently, a conception of the Hadamard composition of the genus $m \geq 1$ was introduced [8]. Moreover, the connection between the growth of the functions and the growth of the Hadamard composition of the genus $m \geq 1$ of F was investigated in the terms of generalized orders and convergence classes. These authors studied the pseudostarlikeness and pseudoconvexity of the Hadamard composition of the genus m. The use of the Hadamard composition of the genus m for Dirichlet series allows us to replace the examination of the growth properties of such a composition by the examination of the growth properties of the dominant function in the composition, etc. Moreover, the approach could be useful in theory of the Dirichlet–Hadamard–Kong product of a finite Dirichlet series [9]. In this product, the exponents of product function, such as the Dirichlet series, are linear combinations of the exponents of generating functions.

Suppose that $\Lambda = (\lambda_n)$ is a sequence of non-negative numbers, $\lambda_0 = 0$, increasing to $+\infty$, and by $S(\Lambda, A)$ we denote a class of Dirichlet series :

$$F(s) = \sum_{n=1}^{\infty} f_n \exp\{s\lambda_n\}, \quad s = \sigma + it, \tag{1}$$

with the abscissa of the absolute convergence $\sigma_a = A \in (-\infty, +\infty]$ such that $\overline{\lim\limits_{n\to\infty}}(\ln |f_n| + A\lambda_n) = +\infty$. The abscissa σ_a is some analog of the radius R of convergence; if we let $\lambda_n = n$ and $z = e^s$, then we obtain a power series with $R = e^{\sigma_A}$.

For $\sigma < A$, let $M_F(\sigma) = \sup\{|F(\sigma + it)| : t \in \mathbb{R}\}$ and $\mu_F(\sigma) = \max\{|f_n| \exp\{\lambda_n \sigma\} : n \geq 0\}$ be the maximal term of the series (1). If $-\infty < A < +\infty$, then the function $\mu_F(\sigma)$ can be bounded on $(-\infty, A)$, and in order that $\mu_F(\sigma) \uparrow +\infty$ as $\sigma \uparrow A$, it is necessary and sufficient that $\overline{\lim\limits_{n\to\infty}}(\ln |f_n| + A\lambda_n) = +\infty$. In what follows, we will assume that $(\ln |f_n| + A\lambda_n) \to +\infty$ as $n \to \infty$. Let us prove $M_F(\sigma) \geq \mu_F(\sigma)$:

$$\begin{aligned}\frac{1}{2T}\int_{-T}^{T} F(\sigma+it)e^{-(\sigma+it)\lambda_n}dt &= \frac{1}{2T}\int_{-T}^{T}\sum_{m=0}^{\infty} f_m e^{(\sigma+it)\lambda_m}\cdot e^{-(\sigma+it)\lambda_n}dt \\ &= \frac{1}{2T}\int_{-T}^{T}\left(\sum_{m=0}^{n-1} f_m e^{(\sigma+it)(\lambda_m-\lambda_n)} + f_n + \sum_{m=n+1}^{\infty} f_m e^{(\sigma+it)(\lambda_m-\lambda_n)}\right)dt.\end{aligned}$$

The last series uniformly converges in $t\mathbb{R}$. We can integrate it and use such an equality:

$$\frac{1}{2T}\int_{-T}^{T} e^{x(\sigma+it)}dt = \frac{e^{x\sigma}}{ixT}(e^{ixT} - e^{-ixT}) \to 0 \text{ as } T \to +\infty, x \neq 0.$$

Then, we obtain

$$f_n e^{\sigma\lambda_n} = \lim_{T\to+\infty}\frac{1}{2T}\int_{-T}^{T} F(\sigma+it)e^{-it\lambda_n}dt,$$

then

$$\begin{aligned}\mu_F(\sigma) = \max\{|f_n|\exp\{\lambda_n\sigma\} : n \geq 0\} &= \max_{n\geq 0}\left|\lim_{T\to+\infty}\frac{1}{2T}\int_{-T}^{T} F(\sigma+it)e^{-it\lambda_n}dt\right| \\ &\leq \lim_{T\to+\infty}\frac{1}{2T}\int_{-T}^{T}|F(\sigma+it)|dt \leq M_F(\sigma).\end{aligned}$$

In view of the inequality $M_F(\sigma) \geq \mu_F(\sigma)$, the function $M_F(\sigma)$ is increasing to $+\infty$ and continuous on $(-\infty,\ A)$ for each function $F \in S(\Lambda, A)$. Therefore, there exists the function $M_F^{-1}(x)$ inverse to $M_F(\sigma)$, which increases to A on $(|a_0|,\ +\infty)$.

By L, we denote a class of continuous non-negative $(-\infty,\ +\infty)$ functions α such that $\alpha(x) = \alpha(x_0) \geq 0$ for $x \leq x_0$, and $\alpha(x) \uparrow +\infty$ strictly increases to $+\infty$ as $x_0 \leq x \to +\infty$. We say that $\alpha \in L^0$ if $\alpha \in L$ and $\alpha((1+o(1))x) = (1+o(1))\alpha(x)$, as $x \to +\infty$. Finally, $\alpha \in L_{si}$ if $\alpha \in L$ and $\alpha(cx) = (1+o(1))\alpha(x)$ as $x \to +\infty$ for each positive real constant $c \in (0,\ +\infty)$, i. e., α, is a slowly increasing function. Clearly, $L_{si} \subset L^0$.

If $\alpha \in L$, $\beta \in L$, $F \in S(\Lambda, +\infty)$, $G \in S(\Lambda, +\infty)$ and

$$G(s) = \sum_{n=1}^{\infty} g_n \exp\{s\lambda_n\}, \tag{2}$$

then the growth of the function F with respect to the function G is comparable [10,11] to the growth of the function $M_G^{-1}(M_F(\sigma))$ as $\sigma \to +\infty$, i.e., the generalized (α, β)-order $\varrho_{\alpha,\beta}[F]_G$ and the generalized lower (α, β)-order $\lambda_{\alpha,\beta}[F]_G$ of the function $F \in S(\Lambda, +\infty)$ with respect to a function $G \in S(\Lambda, +\infty)$, which we define as follows

$$\varrho_{\alpha,\beta}[F]_G := \overline{\lim_{\sigma\to+\infty}} \frac{\alpha(M_G^{-1}(M_F(\sigma)))}{\beta(\sigma)}, \ \lambda_{\alpha,\beta}[F]_G := \underline{\lim_{\sigma\to+\infty}} \frac{\alpha(M_G^{-1}(M_F(\sigma)))}{\beta(\sigma)}.$$

The connection between the growth of the function $M_G^{-1}(M_F(\sigma))$ and the growth of the functions $M_G(\sigma))$ and $M_F(\sigma)$ in terms of generalized orders has been studied in [10,11], where formulae were found for calculating $\varrho_{\alpha,\beta}[F]_G$ and $\lambda_{\alpha,\beta}[F]_G$ in terms of the coefficients f_n and g_n.

Another approach to studying the growth of the Dirichlet series (1) is to compare the growth of the function $\ln M_F(\sigma)$ with the growth of some convex function $\Phi(\sigma)$. Using the function $\Phi(\sigma)$, we will study the relative growth of a function $F \in S(\Lambda, A)$ with respect to the functions $G \in S(\Lambda, +\infty)$ and $G \in S(\Lambda, 0)$.

2. Relative Φ-Type and Convergence Φ-Class

For $A \in (-\infty, +\infty]$, we denote by $\Omega(A)$ a class of positive unbounded $(-\infty, A)$ functions Φ such that its derivative Φ' is a positive, continuously differentiable, and increasing to $+\infty$ function on $(-\infty, A)$. For example, the function $\Phi(x) = \frac{1}{A-x}$ belongs to the class $\Omega(A)$. Let φ be the inverse function to Φ' and let the function $\Psi(\sigma) = \sigma - \frac{\Phi(\sigma)}{\Phi'(\sigma)}$ be the function associated with Φ in the sense of Newton. Then, according to [12,13], such a defined function, Ψ, is continuously differentiable and increasing to $+\infty$ on $(-\infty, A)$, and the function φ is continuously differentiable and increasing to A on $(0, +\infty)$.

Definition 1 ([14])**.** *For a Dirichlet series with an arbitrary abscissa of absolute convergence $A \in (-\infty, +\infty]$ and for the function $\Phi \in \Omega(A)$, the quantity $T_\Phi[F] = \overline{\lim_{\sigma\uparrow A}} \frac{\ln M_F(\sigma)}{\Phi(\sigma)}$ is called the Φ-type of the function F. By analogy, if $G \in S(\Lambda, +\infty)$ and $F \in S(\Lambda, A)$, $A \in (-\infty, +\infty]$, then we call the quantity*

$$T_\Phi[F]_G = \overline{\lim_{\sigma\uparrow A}} \frac{M_G^{-1}(M_F(\sigma))}{\Phi(\sigma)} \tag{3}$$

as the Φ-type of the function F with respect to the function G.

Now, suppose that $G \in S(\Lambda, 0)$; then, function $M_G(\sigma)$ is continuous and increasing to $+\infty$ on $(-\infty, 0)$; thus, there exists the function $M_G^{-1}(x) < 0$, which is the inverse of the function $M_G(\sigma)$, and which increases to 0 on $(|g_0|, +\infty)$. Therefore, $\dfrac{1}{|M_G^{-1}(x)|}$ strictly increases to $+\infty$, and we can define the Φ-type of the function F with respect to the function G as follows:

$$T_\Phi^0[F]_G = \overline{\lim_{\sigma\uparrow A}} \frac{1}{\Phi(\sigma)|M_G^{-1}(M_F(\sigma)|}. \tag{4}$$

If $G \in S(\Lambda, +\infty)$, then we define

$$T_\Phi[\mu_F]_G = \overline{\lim_{\sigma\uparrow A}} \frac{M_G^{-1}(\mu_F(\sigma))}{\Phi(\sigma)},$$

and if $G \in S(\Lambda, 0)$, then we define

$$T_\Phi^0[\mu_F]_G = \overline{\lim_{\sigma\uparrow A}} \frac{1}{\Phi(\sigma)|M_G^{-1}(\mu_F(\sigma)|}.$$

Above, we have proved that $\mu_F(\sigma) \le M_F(\sigma)$. Obviously, M_G^{-1} is an increasing function. Then, for all $\sigma < A$, one has $M_G^{-1}(\mu_F(\sigma)) \le M_G^{-1}(M_F(\sigma))$; that is, $\frac{M_G^{-1}(\mu_F(\sigma))}{\Phi(\sigma)} \le \frac{M_G^{-1}(M_F(\sigma))}{\Phi(\sigma)}$. This means that $T_\Phi[\mu_F]_G \le T_\Phi[F]_G$. Similarly, $T^0_\Phi[\mu_F]_G \le T^0_\Phi[F]_G$. To obtain estimates for $T_\Phi[F]_G$ and $T^0_\Phi[F]_G$ from above, we need the following lemma.

Lemma 1 ([15]). *Let $F \in S(\Lambda, A)$, $A \in (-\infty, +\infty]$. Suppose that a function f is positive, continuous, and increasing to A on $(-\infty, A)$. For $\sigma < A$, we assert that*

$$p(\sigma) = \sup\left\{\frac{\sigma - t}{f(\sigma) - f(t)} : \sigma_0 \le t < \sigma\right\}$$

and let g be a function continuous on $(-\infty, +\infty)$ such that $g(x) = f^{-1}(x)$ on $(-\infty, A)$ and $g(x) = A$ for $x \ge A$ if $A < +\infty$.

If

$$\sum_{n=1}^{\infty} |f_n| \exp\left\{\lambda_n g\left(\frac{1}{\lambda_n} \ln \frac{1}{|f_n|}\right)\right\} \le K_0 < +\infty, \tag{5}$$

then for all $\sigma < A$,

$$M_F(\sigma) \le K_0(\mu_F(f(\sigma))^{p(\sigma)} + K_0 + |f_0|. \tag{6}$$

Lemma 1 is proved in [15] for the case $A = +\infty$ and in [16] for the case $-\infty < A \le +\infty$. Using this lemma, we prove the following statement.

Lemma 2. *Let $F \in S(\Lambda, A)$, $A \in (-\infty, +\infty]$, $G \in S(\Lambda, +\infty)$ (or $G \in S(\Lambda, 0)$), and $M_G^{-1}(e^x) \in L_{si}$ (respectively, $(1/|M_G^{-1}(e^x)|) \in L_{si}$). Suppose that $\Phi \in \Omega(A)$,*

$$Q_\Phi := \varlimsup_{\sigma \uparrow A} \frac{\Phi(\sigma)}{\Phi(\Psi(\sigma))} < +\infty$$

m and for all $\sigma \in [a, A]$,

$$0 < h \le \frac{\Phi''(\sigma)\Phi(\sigma)}{(\Phi'(\sigma))^2} \le H < +\infty. \tag{7}$$

If

$$\sum_{n=1}^{\infty} |f_n| \exp\left\{\lambda_n \Psi\left(\frac{1}{\lambda_n} \ln \frac{1}{|f_n|}\right)\right\} \le K_0 < +\infty, \tag{8}$$

then $T_\Phi[F]_G \le Q_\Phi T_\Phi[\mu_F]_G$ (respectively, $T^0_\Phi[F]_G \le Q_\Phi T^0_\Phi[\mu_F]_G$).

Proof. Choose $f(\sigma) = \Psi^{-1}(\sigma)$ in Lemma 1. Then, $g(\sigma) = \Psi(\sigma)$ and (8) implies (5). Since $\Psi'(\sigma) = \dfrac{\Phi''(\sigma)\Phi(\sigma)}{(\Phi'(\sigma))^2}$, condition (7) implies $0 < h \le \Psi'(\sigma) \le H < +\infty$; therefore,

$$p(\sigma) = \sup\left\{\frac{\Psi(\sigma) - \Psi(t)}{\sigma - t} : \Psi^{-1}(\sigma_0) \le t < \Psi^{-1}(\sigma)\right\}$$

$$= \sup\{\Psi'(\xi) : \Psi^{-1}(\sigma_0) \le t < \xi < \Psi^{-1}(\sigma)\} \le H,$$

and by Lemma 1, for all $\sigma < A$ that are sufficiently close to A, the following two-sided inequality holds:

$$M_F(\sigma) \le K_0\Big(\mu_F(\Psi^{-1}(\sigma))\Big)^H + K_0 + |f_0| \le \Big(\mu_F(\Psi^{-1}(\sigma))\Big)^{H+1} \tag{9}$$

because $\mu_F(\sigma) \to +\infty$ as $\sigma \uparrow A$.

If $G \in S(\Lambda, +\infty)$ and $M_G^{-1}(e^x) \in L_{si}$, then, from (9), we obtain

$$T_\Phi[F]_G \le \varlimsup_{\sigma \uparrow A} \frac{M_G^{-1}((\mu_F(\Psi^{-1}(\sigma))^{H+1})}{\Phi(\sigma)} = \varlimsup_{\sigma \uparrow A} \frac{M_G^{-1}((\exp\{(H+1)\ln \mu_F(\Psi^{-1}(\sigma)\}))}{\Phi(\sigma)}$$

$$= \varlimsup_{\sigma \uparrow A} \frac{(1+o(1))M_G^{-1}(\exp\{\ln \mu_F(\Psi^{-1}(\sigma)\})}{\Phi(\sigma)} = \varlimsup_{\sigma \uparrow A} \frac{M_G^{-1}(\mu_F(\Psi^{-1}(\sigma))}{\Phi(\sigma)}$$

$$= \varlimsup_{\sigma \uparrow A} \frac{M_G^{-1}(\mu_F(\Psi^{-1}(\sigma))}{\Phi(\Psi^{-1}(\sigma))} \frac{\Phi(\Psi^{-1}(\sigma))}{\Phi(\sigma)} \le \varlimsup_{\sigma \uparrow A} \frac{M_G^{-1}(\mu_F(\sigma))}{\Phi(\sigma)} \varlimsup_{\sigma \uparrow A} \frac{\Phi(\sigma)}{\Phi(\Psi(\sigma))} = T_\Phi[\mu_F]_G Q_\Phi.$$

Similarly, If $G \in S(\Lambda, 0)$ and $(1/|M_G^{-1}(e^x)|) \in L_{si}$ then

$$T_\Phi^0[F]_G \le \varlimsup_{\sigma \uparrow A} \frac{1}{\Phi(\sigma)|M_G^{-1}((\mu_F(\Psi^{-1}(\sigma))^{H+1})|} = \varlimsup_{\sigma \uparrow A} \frac{1}{\Phi(\sigma)|M_G^{-1}(\mu_F(\Psi^{-1}(\sigma))|}$$

$$= \varlimsup_{\sigma \uparrow A} \frac{1}{\Phi(\Psi^{-1}(\sigma))|M_G^{-1}(\mu_F(\Psi^{-1}(\sigma))|} \frac{\Phi(\Psi^{-1}(\sigma))}{\Phi(\sigma)} \le T_\Phi^0[\mu_F]_G Q_\Phi.$$

The proof of Lemma 2 is completed. □

In the case when the function $F \in S(\Lambda, A)$, $A \in (-\infty, +\infty]$ is of the Φ-type zero with $\Phi \in \Omega(A)$ for the study of the growth of $\ln M_F(\sigma)$, the authors of paper [14] introduced the convergence Φ-class on the condition that the following integral; i.e., $\int_{\sigma_0}^{A} \frac{\Phi'(\sigma)\ln M_F(\sigma)}{\Phi^2(\sigma)} d\sigma$, is finite.

Definition 2. *Similarly, we will say that a function $F \in S(\Lambda, A)$ belongs to the convergence Φ-class with respect to the function $G \in S(\Lambda, +\infty)$ if*

$$\int_{\sigma_0}^{A} \frac{\Phi'(\sigma) M_G^{-1}(M_F(\sigma))}{\Phi^2(\sigma)} d\sigma < +\infty, \tag{10}$$

and that it belongs to the convergence Φ-class with respect to the function $G \in S(\Lambda, 0)$ if

$$\int_{\sigma_0}^{A} \frac{\Phi'(\sigma)}{\Phi^2(\sigma)|M_G^{-1}(M_F(\sigma))|} d\sigma < +\infty. \tag{11}$$

In Section 5 we present examples of functions F belonging to the convergence Φ-class with the respect to the function G.

Lemma 3. *Let $A \in (-\infty, +\infty]$, $F \in S(\Lambda, A)$, $\Phi \in \Omega(A)$, and $\Phi(\sigma) = O(\Phi(\Psi(\sigma))$ as $\sigma \uparrow A$ and conditions (7) and (8) hold. Suppose that $M_G^{-1}(e^x) \in L^0$ if $G \in S(\Lambda, +\infty)$, and $(1/|M_G^{-1}(e^x)|) \in L^0$ if $G \in S(\Lambda, 0)$. In order for F to belong to the convergence Φ-class with respect to $G \in S(\Lambda, +\infty)$, it is necessary and sufficient that*

$$\int_{\sigma_0}^{A} \frac{\Phi'(\sigma) M_G^{-1}(\mu_F(\sigma))}{\Phi^2(\sigma)} d\sigma < +\infty. \tag{12}$$

In order for F to belong to the convergence Φ-class with respect to $G \in S(\Lambda, 0)$, it is necessary and sufficient that

$$\int_{\sigma_0}^{A} \frac{\Phi'(\sigma)}{\Phi^2(\sigma)|M_G^{-1}(\mu_F(\sigma))|} d\sigma < +\infty. \tag{13}$$

Proof. In view of Cauchy's inequality, the finiteness of the integral (10) implies the finiteness of the integral (12). Similarly, the validity of (11) yields the validity of (12). Before moving on to the proof of the converse implications, we remark that it is proved in [17] that if $\beta \in L^0$, then

$$\overline{\lim_{x\to+\infty}} \beta((1+\varepsilon)x)/\beta(x) = A(\varepsilon) \downarrow 1 \text{ as } \varepsilon \downarrow 0 \text{ and}$$

$$\overline{\lim_{x\to+\infty}} \beta(cx)/\beta(x) = B = B(c) < +\infty \text{ for } c = \text{const} > 0.$$

We also remark that the condition $\Phi(\sigma) = O(\Phi(\Psi(\sigma))$ as $\sigma \uparrow A$ implies $\Phi(\sigma) \le Q\Phi(\Psi(\sigma))$ for all $\sigma < A$, where $Q < +\infty$.

Therefore, estimate (9) implies

$$\int_{\sigma_0}^{A} \frac{\Phi'(\sigma)M_G^{-1}(M_F(\sigma))}{\Phi^2(\sigma)}d\sigma \le \int_{\sigma_0}^{A} \frac{\Phi'(\sigma)M_G^{-1}((\exp\{(H+1)\ln \mu_F(\Psi^{-1}(\sigma)\}))}{\Phi^2(\sigma)}d\sigma$$

$$\le B\int_{\sigma_0}^{A} \frac{\Phi'(\sigma)M_G^{-1}(\mu_F(\Psi^{-1}(\sigma))}{\Phi^2(\sigma)}d\sigma \le B\int_{\sigma_0}^{A} \frac{\Phi'(\Psi^{-1}(\sigma))M_G^{-1}(\mu_F(\Psi^{-1}(\sigma))}{\Phi^2(\sigma)}d\sigma$$

$$= B\int_{\sigma_0}^{A} \frac{\Phi'(\Psi^{-1}(\sigma))M_G^{-1}(\mu_F(\Psi^{-1}(\sigma))}{\Phi^2(\Psi^{-1}(\sigma))}\frac{\Phi^2(\Psi^{-1}(\sigma))}{\Phi^2(\sigma)}d\sigma$$

$$\le BQ^2\int_{\sigma_0}^{A} \frac{\Phi'(\Psi^{-1}(\sigma))M_G^{-1}(\mu_F(\Psi^{-1}(\sigma))}{\Phi^2(\Psi^{-1}(\sigma))}d\sigma = BQ^2\int_{\Psi^{-1}(\sigma_0)}^{A} \frac{\Phi'(\sigma)M_G^{-1}(\mu_F(\sigma))}{\Phi^2(\sigma)}\Psi'(\sigma)d\sigma$$

$$\le BQ^2H\int_{\Psi^{-1}(\sigma_0)}^{A} \frac{\Phi'(\sigma)M_G^{-1}(\mu_F(\sigma))}{\Phi^2(\sigma)}d\sigma$$

and thus, (12) implies (10).

Similarly,

$$\int_{\sigma_0}^{A} \frac{\Phi'(\sigma)}{\Phi^2(\sigma)|M_G^{-1}(M_F(\sigma))|}d\sigma \le B\int_{\sigma_0}^{A} \frac{\Phi'(\Psi^{-1}(\sigma))}{|M_G^{-1}(\mu_F(\Psi^{-1}(\sigma))|\Phi^2(\sigma)}d\sigma$$

$$\le BQ^2\int_{\Psi^{-1}(\sigma_0)}^{A} \frac{\Phi'(\sigma)}{|M_G^{-1}(\mu_F(\sigma))|\Phi^2(\sigma)}\Psi'(\sigma)d\sigma \le BQ^2H\int_{\Psi^{-1}(\sigma_0)}^{A} \frac{\Phi'(\sigma)}{|M_G^{-1}(\mu_F(\sigma))|\Phi^2(\sigma)}d\sigma$$

and thus, (13) implies (11). The proof of Lemma 3 is thus completed. □

3. Φ-Type of Hadamard Compositions

Below, we introduce the notion of the Hadamard composition of genus m for the Dirichlet series. It was first introduced in [8] for the Dirichlet series in the half-plane. The multidimensional Hadamard composition was considered in [18].

Definition 3 ([8])**.** *Dirichlet series* (1) *is called the Hadamard composition of genus m of the following Dirichlet series*

$$F_j(s) = \sum_{n=1}^{\infty} f_{n,j}\exp\{s\lambda_n\}, \quad 1 \le j \le p \tag{14}$$

if $f_n = P(f_{n,1}, ..., f_{n,p})$, *where* $P(x_1, ..., x_p) = \sum\limits_{k_1+\dots+k_p=m} c_{k_1...k_p} x_1^{k_1} \cdot ... \cdot x_p^{k_p}$ *is a homogeneous polynomial of degree* $m \geq 1$.

We remark that the usual Hadamard composition [19,20] is a special case of the Hadamard composition of the genus $m = 2$. The quasi-Hadamard product was considered in [21].

It is clear that if the function F is the Hadamard composition of genus $m \geq 1$ of the functions F_j, then

$$|f_n| \leq \sum_{k_1+\dots+k_p=m} |c_{k_1...k_p}||f_{n,1}|^{k_1} \cdot ... \cdot |f_{n,p}|^{k_p}. \tag{15}$$

The function F_1 is called dominant, if $|c_{m0...0}||f_{n,1}|^m \neq 0$ and $|f_{n,j}| = o(|f_{n,1}|)$ as $n \to \infty$ for $2 \leq j \leq p$. It is shown in [8] that if the function F_1 is dominant then

$$|f_n| = (1 + o(1))|c_{m0...0}||f_{n,1}|^m, \quad n \to \infty. \tag{16}$$

For the Hadamard composition of Dirichlet series (14), the following theorem is true.

Theorem 1. *Let* $A \in (-\infty, +\infty]$, $\Phi \in \Omega(A)$, $Q_\Phi < +\infty$ *and conditions* (7) *and* (8) *hold. Let* $G \in S(\Lambda, +\infty)$, $M_G^{-1}(e^x) \in L_{si}$, *and the function* $F \in S(\Lambda, A)$ *is the Hadamard composition of genus* $m \geq 2$ *of the functions* $F_j \in S(\Lambda, A)$, $1 \leq j \leq p$.

If $T_\Phi = \max\{T_\Phi[F_j]_G : 1 \leq j \leq p\} < +\infty$ *and either* $A > 0$ *or* $A \leq 0$ *and*

$$\overline{\lim_{\sigma\uparrow A}} \frac{\ln M_G(T\Phi(\sigma))}{\ln M_G(T\Phi(m\sigma))} = v(m) < +\infty$$

for $T > T_\Phi$, *then* $T_\Phi[F]_G \leq Q_\Phi T_\Phi$.

If, in addition, F_1 *is dominant, then* $T_\Phi[F]_G \geq T_\Phi[F_1]_G / Q_\Phi$ *if* $A \leq 0$ *and* $T_\Phi[F]_G \geq T_\Phi[F_1]_G / (Q_\Phi P_\Phi(m))$ *if* $A > 0$ *and* $\overline{\lim\limits_{\sigma\uparrow A}} \Phi(m\sigma)/\Phi(\sigma) = P_\Phi(m) < +\infty$.

Proof. Since (15) implies

$$|f_n|e^{m\sigma\lambda_n} \leq \sum_{k_1+\dots+k_p=m} |c_{k_1...k_p}|(|f_{n,1}|e^{\sigma\lambda_n})^{k_1} \cdot ... \cdot (|f_{n,p}|e^{\sigma\lambda_n})^{k_p},$$

we have

$$\mu_F(m\sigma) \leq \sum_{k_1+\dots+k_p=m} |c_{k_1...k_p}|\mu_{F_1}(\sigma)^{k_1} \cdot ... \cdot \mu_{F_p}(\sigma)^{k_p}. \tag{17}$$

In view of (3) and Cauchy's inequality, we have

$$\frac{M_G^{-1}(\mu_{F_j}(\sigma))}{\Phi(\sigma)} \leq T$$

for every $T > T_\Phi$, all $\sigma \in [\sigma_0, A)$, and all j, i.e., $\mu_{F_j}(\sigma) \leq M_G(T\Phi(\sigma))$, and from (17), we obtain for all $\sigma \in [\sigma_0, A)$

$$\mu_F(m\sigma) \leq C_1 M_G^m(T\Phi(\sigma)), \quad C_1 = \sum_{k_1+\dots+k_p=m} |c_{k_1...k_p}|. \tag{18}$$

If $A > 0$, then $\mu_F(m\sigma) \geq \mu_F(\sigma)$ for all $\sigma \in (0, A)$, and (18) implies $\mu_F(\sigma) \leq C_1 M_G^m (T\Phi(\sigma))$. Therefore, in view of the condition $M_G^{-1}(e^x) \in L_{si}$, we obtain

$$T_\Phi[\mu_F]_G \leq \overline{\lim_{\sigma\uparrow A}} \frac{M_G^{-1}(\exp\{m \ln M_G(T\Phi(\sigma)) + \ln C_1\})}{\Phi(\sigma)} = T,$$

i.e., in view of the arbitrariness of T, we obtain $T_\Phi[\mu_F]_G \le T_\Phi$. On the other hand, through Lemma 2, one has $T_\Phi[F]_G \le Q_\Phi T_\Phi[\mu_F]_G$. Therefore, $T_\Phi[F]_G \le Q_\Phi T_\Phi$.

Now, let $A \le 0$, and the inequality

$$\varlimsup_{\sigma\uparrow A} \frac{\ln M_G(T\Phi(\sigma))}{\ln M_G(T\Phi(m\sigma))} = v(m) < +\infty$$

is true. Then, $M_G(T\Phi(\sigma)) \le M_G^v(T\Phi(m\sigma))$ for every $v > v(m)$ and all $\sigma \in [\sigma_0(v), A)$, and (18) implies $\mu_F(m\sigma) \le C_1 M_G^{mv}(T\Phi(m\sigma))$, whence, as above, in view of the condition $M_G^{-1}(e^x) \in L_{si}$, we obtain

$$T_\Phi[\mu_F]_G \le \varlimsup_{\sigma\uparrow A} \frac{M_G^{-1}(\mu_F(m\sigma))}{\Phi(m\sigma)} \le \varlimsup_{\sigma\uparrow A} \frac{M_G^{-1}(C_1 M_G^{mv}(T\Phi(m\sigma))}{\Phi(m\sigma)} = T,$$

i.e., in view of the arbitrariness of T and Lemma 2, $T_\Phi[F]_G \le Q_\Phi T_\Phi$, Q.E.D.

If F_1 is dominant, then (16) implies $c_1\mu_{F_1}(\sigma)^m \le \mu_F(m\sigma) \le c_2\mu_{F_1}(\sigma)^m$. Therefore, if $A \le 0$, then $\mu_F(\sigma) \ge c_1\mu_{F_1}(\sigma)^m$ for $\sigma < A$, and in view of the condition $M_G^{-1}(e^x) \in L_{si}$

$$T_\Phi[\mu_F]_G \ge \varlimsup_{\sigma\uparrow A} \frac{M_G^{-1}(c_1\mu_{F_1}(\sigma)^m)}{\Phi(\sigma)} = \varlimsup_{\sigma\uparrow A} \frac{M_G^{-1}(\mu_{F_1}(\sigma))}{\Phi(\sigma)} = T_\Phi[\mu_{F_1}]_G$$

and through Lemma 2, we have $T_\Phi[F]_G \ge T_\Phi[\mu_F]_G \ge T_\Phi[\mu_{F_1}]_G \ge T_\Phi[F_1]_G/Q_\Phi$.

If $A > 0$ and $\varlimsup\limits_{\sigma\uparrow A} \Phi(m\sigma)/\Phi(\sigma) = P_\Phi(m) < +\infty$, then, similarly, one has

$$T_\Phi[\mu_F]_G = \varlimsup_{\sigma\uparrow A} \frac{M_G^{-1}(\mu_F(m\sigma))}{\Phi(m\sigma)} \ge \varlimsup_{\sigma\uparrow A} \frac{M_G^{-1}(c_1\mu_{F_1}(\sigma)^m)}{\Phi(m\sigma)}$$

$$= \varlimsup_{\sigma\uparrow A} \frac{M_G^{-1}(\mu_{F_1}(\sigma))}{\Phi(\sigma)} \frac{\Phi(\sigma)}{\Phi(m\sigma)} \ge \varlimsup_{\sigma\uparrow A} \frac{M_G^{-1}(\mu_{F_1}(\sigma))}{\Phi(\sigma)} \varliminf_{\sigma\uparrow A} \frac{\Phi(\sigma)}{\Phi(m\sigma)} = \frac{T_\Phi[\mu_{F_1}]_G}{P_\Phi(m),}$$

and thus, through Lemma 2,

$$T_\Phi[F]_G \ge T_\Phi[\mu_F]_G \ge \frac{T_\Phi[\mu_{F_1}]_G}{P_\Phi(m)} \ge \frac{T_\Phi[F_1]_G}{Q_\Phi P_\Phi(m)}.$$

The proof of Theorem 1 is thus completed. □

Let us now consider the case where $m = 1$; i.e., $f_n = c_1 f_{n,1} + \cdots + c_p f_{n,p}$. Then, $\mu_F(\sigma) \le |c_1|\mu_{F_1}(\sigma) + \cdots + |c_p|\mu_{F_p}(\sigma)$, whence we obtain $T_\Phi[\mu_F]_G \le T_\Phi = \max\{T_\Phi[\mu_{F_j}]_G : 1 \le j \le p\}$, because $M_G^{-1}(x) \in L_{si}$. If F_1 is dominant, then $\mu_F(\sigma) \asymp \mu_{F_1}(\sigma)$ and $T_\Phi[\mu_F]_G = T_\Phi[\mu_{F_1}]_G$. Therefore, Lemma 2 implies the following statement.

Corollary 1. *Let $A \in (-\infty, +\infty]$, $\Phi \in \Omega(A)$ and conditions (7) and (8) hold. Let $G \in S(\Lambda, +\infty)$, and the function $F \in S(\Lambda, A)$ is the Hadamard composition of genus $m = 1$ of the function $F_j \in S(\Lambda, A)$, $1 \le j \le p$. Then, $T_\Phi[F]_G \le Q_\Phi \max\{T_\Phi[F_j]_G : 1 \le j \le p\}$. If, in addition, F_1 is a dominant, then*

$$T_\Phi[F_1]_G/Q_\Phi \le T_\Phi[F]_G \le Q_\Phi T_\Phi[F_1]_G.$$

In Theorem 1 and Corollary 1, we assumed that the comparing function G belongs to the class $S(\Lambda, +\infty)$. Now, we consider the case $G \in S(\Lambda, 0)$.

Theorem 2. *Let $A \in (-\infty, +\infty]$, $\Phi \in \Omega(A)$, $Q_\Phi < +\infty$, and conditions (7) and (8) are fulfilled. Let $G \in S(\Lambda, 0)$, $(1/|M_G^{-1}(e^x)|) \in L_{si}$, and the function $F \in S(\Lambda, A)$ is the Hadamard composition of genus $m \ge 2$ of the function $F_j \in S(\Lambda, A)$, $1 \le j \le p$.*

If $T_\Phi^0 = \max\{T_\Phi^0[F_j]_G : 1 \le j \le p\} < +\infty$ *and either* $A > 0$ *or* $A \le 0$ *and*

$$\overline{\lim_{\sigma\uparrow A}} \frac{\ln M_G(-1/(T\Phi(\sigma)))}{\ln M_G(-1/(T\Phi(m\sigma)))} = v(m) < +\infty$$

for $T > T_\Phi^0$*, then* $T_\Phi^0[F]_G \le Q_\Phi T_\Phi^0$*.*

If, in addition, F_1 *is dominant, then* $T_\Phi^0[F]_G \ge T_\Phi^0[F_1]_G/Q_\Phi$ *if* $A \le 0$*, and* $T_\Phi[F]_G \ge T_\Phi[F_1]_G/(Q_\Phi P_\Phi(m))$ *if* $A > 0$ *and* $\overline{\lim\limits_{\sigma\uparrow A}}\, \Phi(m\sigma)/\Phi(\sigma) = P_\Phi(m) < +\infty$*.*

Proof. In view of (4), we have $\dfrac{1}{|M_G^{-1}(\mu_{F_j}(\sigma))|\Phi(\sigma)} \le T$ for every $T > T_\Phi^0$, all $\sigma \in [\sigma_0, A)$ and all j, i.e., $\mu_{F_j}(\sigma) \le M_G\left(-\dfrac{1}{T\Phi(\sigma)}\right)$, and from (17), we obtain for all $\sigma \in [\sigma_0, A)$

$$\mu_F(m\sigma) \le C_1 M_G^m\left(-\frac{1}{T\Phi(\sigma)}\right), \quad C_1 = \sum_{k_1+\dots+k_p=m} |c_{k_1\dots k_p}|. \tag{19}$$

If $A > 0$, then (19) implies $\mu_F(\sigma) \le C_1 M_G^m(-1/(T\Phi(\sigma)))$ for $0 < \sigma_0(T) \le \sigma < A)$. Therefore, in view of the condition $(1/|M_G^{-1}(e^x)|) \in L_{si}$, we obtain

$$T_\Phi[\mu_F]_G \le \overline{\lim_{\sigma\uparrow A}} \frac{1}{\left|M_G^{-1}(C_1 M_G^m(-1/(T\Phi(\sigma))))\right|\Phi(\sigma)}$$

$$= \overline{\lim_{\sigma\uparrow A}} \frac{1}{\left|M_G^{-1}(\exp\{m \ln M_G(-1/(T\Phi(\sigma))) + \ln C_1\})\right|\Phi(\sigma)}$$

$$= \overline{\lim_{\sigma\uparrow A}} \frac{1}{|-1/(T\Phi(\sigma))|\Phi(\sigma)} = T,$$

i.e., in view of the arbitrariness of T, we obtain $T_\Phi^0[\mu_F]_G \le T_\Phi^0$. On the other hand, through Lemma 2, the following inequality $T_\Phi^0[F]_G \le Q_\Phi T_\Phi^0[\mu_F]_G$ holds. Therefore, $T_\Phi^0[F]_G \le Q_\Phi T_\Phi^0$.

Suppose that $A \le 0$ and

$$\overline{\lim_{\sigma\uparrow A}} \frac{\ln M_G(-1/(T\Phi(\sigma)))}{\ln M_G(-1/(T\Phi(m\sigma)))} = v(m) < +\infty,$$

then

$$M_G(-1/(T\Phi(\sigma))) \le M_G^v(-1/(T\Phi(m\sigma)))$$

for every $v > v(m)$ and all $\sigma \in [\sigma_0(v), A)$. Multiplying the last estimate m times by itself and applying (19), we deduce $\mu_F(m\sigma) \le C_1 M_G^{mv}(-1/(T\Phi(m\sigma)))$. Hence, as above, in view of the condition $(1/|M_G^{-1}(e^x)|) \in L_{si}$, we obtain $T_\Phi^0[\mu_F]_G \le T$, i.e., in view of the arbitrariness of T and Lemma 2, one has $T_\Phi^0[F]_G \le Q_\Phi T_\Phi^0$, Q.E.D.

If the function F_1 is a dominant and $A \le 0$, then (16) implies $\mu_F(\sigma) \ge \mu_F(m\sigma) \ge c_1\mu_{F_1}(\sigma)^m$ for $\sigma < A$, and in view of the condition $(1/|M_G^{-1}(e^x)|) \in L_{si}$, as above, we obtain $T_\Phi^0[\mu_F]_G \ge T_\Phi^0[\mu_{F_1}]_G$, and through Lemma 2, we have $T_\Phi^0[F]_G \ge T_\Phi^0[F_1]_G/Q_\Phi$.

If the function F_1 is a dominant, $A > 0$ and $\overline{\lim\limits_{\sigma\uparrow A}}\, \Phi(m\sigma)/\Phi(\sigma) = P_\Phi(m) < +\infty$, then, as in the proof of Theorem 1, we obtain $T_\Phi^0[\mu_F]_G \ge T_\Phi^0[\mu_{F_1}]_G/P_\Phi(m)$ and by Lemma 2 we have $T_\Phi^0[F]_G \ge T_\Phi^0[F_1]_G/(Q_\Phi P_\Phi(m))$. The proof of Theorem 2 is thus completed. □

Repeating the proof of Corollary 1, we obtain the following statement.

Corollary 2. *Let* $A \in (-\infty, +\infty]$, $\Phi \in \Omega(A)$ *and conditions* (7) *and* (8) *hold. Let* $G \in S(\Lambda, 0)$, $(1/|M_G^{-1}(x)|) \in L_{si}$ *, and the function* $F \in S(\Lambda, A)$ *is a Hadamard composition of genus* $m = 1$

of the function $F_j \in S(\Lambda, A)$, $1 \le j \le p$. *Then,* $T^0_\Phi[F]_G \le Q_\Phi \max\{T^0_\Phi[F_j]_G : 1 \le j \le p\}$. *If, in addition,* F_1 *is dominant, then* $T^0_\Phi[F_1]_G / Q_\Phi \le T^0_\Phi[F]_G \le Q_\Phi T^0_\Phi[F_1]_G$.

4. Convergence Φ-Classes of the Hadamard Compositions

Let, at first, $G \in S(\Lambda, +\infty)$, and the function $F \in S(\Lambda, A)$, $A \in (-\infty, +\infty]$ is the Hadamard composition of genus $m \ge 1$ of the functions $F_j \in S(\Lambda, A)$. Suppose that each F_j belongs to the convergence Φ-class with respect to G; i.e.,

$$\int_{\sigma_0}^{A} \frac{\Phi'(\sigma) M_G^{-1}(M_{F_j}(\sigma))}{\Phi^2(\sigma)} d\sigma < +\infty.$$

Since $\mu_{F_j}(\sigma) \uparrow +\infty$, as $\sigma \uparrow A$, we have

$$\begin{aligned} \ln\left(\sum_{k_1+\cdots+k_p=m} |c_{k_1\ldots k_p}| \mu_{F_1}(\sigma)^{k_1} \cdot \ldots \cdot \mu_{F_p}(\sigma)^{k_p}\right) \le \\ \le \sum_{k_1+\cdots+k_p=m} \ln\left(|c_{k_1\ldots k_p}| \mu(\sigma, F_1)^{k_1} \cdot \ldots \cdot \mu(\sigma, F_p)^{k_p}\right) + \ln(m+1) \\ = \sum_{k_1+\cdots+k_p=m} (\ln(|c_{k_1\ldots k_p}| + k_1 \ln \mu(\sigma, F_1) + \ldots + k_p \ln \mu(\sigma, F_p)) + \ln(m+1) \\ = \sum_{k_1+\cdots+k_p=m} (k_1 \ln \mu(\sigma, F_1) + \ldots + k_p \ln \mu(\sigma, F_p)) + C_1, \end{aligned} \tag{20}$$

where $C_1 = \sum_{k_1+\cdots+k_p=m} \ln^+ |c_{k_1\ldots k_p}| + \ln(m+1)$.

Theorem 3. *Let* $A \in (-\infty, +\infty]$, $F \in S(\Lambda, A)$, $\Phi \in \Omega(A)$, $\Phi(\sigma) = O(\Phi(\Psi(\sigma))$, *as* $\sigma \uparrow A$, *and let conditions* (7) *and* (8) *hold. Let* $G \in S(\Lambda, 0)$, $M_G^{-1}(e^x) \in L^0$, *and the function* $F \in S(\Lambda, A)$ *is the Hadamard composition of genus* $m \ge 1$ *of the function* $F_j \in S(\Lambda, A)$, $1 \le j \le p$.

If all functions F_j *belong to the convergence* Φ*-class with respect to* G, *and either* $A > 0$ *or* $A = 0$ *and* $\Phi(\sigma)/\Phi(m\sigma) \le P_m < +\infty$ *for all* $\sigma < 0$ *or* $A < 0$ *and* $m = 1$, *then the function* F *belongs to the convergence* Φ*-class with respect to* G.

If the function F *belongs to the convergence* Φ*-class with respect to* G, *then the function* F_1 *is dominant, and either* $A \le 0$ *or* $A = +\infty$ *and* $\Phi(m\sigma)/\Phi(\sigma) \le p_m < +\infty$ *for all* $\sigma < +\infty$ *or* $0 < A < +\infty$ *and* $m = 1$; *then, all functions* F_j *belong to the convergence* Φ*-class with respect to* G.

Proof. From (17) and (20), we obtain

$$\begin{aligned} M_G^{-1}(\mu_F(m\sigma)) \le M_G^{-1}\left(\sum_{k_1+\cdots+k_p=m} |c_{k_1\ldots k_p}| \mu_{F_1}(\sigma)^{k_1} \cdot \ldots \cdot \mu_{F_p}(\sigma)^{k_p}\right) \\ \le M_G^{-1}\left(\exp\left\{\sum_{k_1+\cdots+k_p=m} (k_1 \ln \mu(\sigma, F_1) + \ldots + k_p \ln \mu(\sigma, F_p)) + C_1\right\}\right) \\ \le M_G^{-1}\big(\exp\{m \max\{\ln \mu(\sigma, F_j) : 1 \le j \le p\} + C_1\}\big) \\ \le K_1 M_G^{-1}\big(\exp\{\max\{\ln \mu(\sigma, F_j) : 1 \le j \le p\}\}\big) \\ = K_1 \max\{M_G^{-1}(\mu(\sigma, F_j)) : 1 \le j \le p\} \\ \le K_1(M_G^{-1}(\mu(\sigma, F_1)) + \cdots + M_G^{-1}(\mu(\sigma, F_p))). \end{aligned}$$

Therefore, if all functions F_j belong to the convergence Φ-class with respect to the function G, then

$$\int_{\sigma_0}^{A} \frac{\Phi'(\sigma)M_G^{-1}(\mu_F(m\sigma))}{\Phi^2(\sigma)}d\sigma \leq$$

$$\leq K_1\left(\int_{\sigma_0}^{A} \frac{\Phi'(\sigma)M_G^{-1}(\mu_{F_1}(\sigma))}{\Phi^2(\sigma)}d\sigma + \cdots + \int_{\sigma_0}^{A} \frac{\Phi'(\sigma)M_G^{-1}(\mu_{F_p}(\sigma))}{\Phi^2(\sigma)}d\sigma\right) < +\infty. \quad (21)$$

If $A > 0$, then $\mu_F(\sigma) \leq \mu_F(m\sigma)$ for $\sigma \in [0, A)$, and (21) implies (12); i.e., through Lemma 3, the function F belongs to the convergence Φ-class with respect to the function G.

If $A = 0$ and $\Phi(\sigma) \leq P_m\Phi(m\sigma)$ for all $\sigma < 0$, then

$$+\infty > \int_{\sigma_0}^{A} \frac{\Phi'(\sigma)M_G^{-1}(\mu_F(m\sigma))}{\Phi^2(\sigma)}d\sigma = \int_{\sigma_0}^{0} \frac{\Phi'(m\sigma)M_G^{-1}(\mu_F(m\sigma))}{\Phi^2(m\sigma)}\frac{\Phi^2(m\sigma)}{\Phi^2(\sigma)}\frac{\Phi'(\sigma)}{\Phi'(m\sigma)}d\sigma$$

$$\geq \frac{1}{mP_m}\int_{\sigma_0}^{0} \frac{\Phi'(m\sigma)M_G^{-1}(\mu_F(m\sigma))}{\Phi^2(m\sigma)}d(m\sigma) = \frac{1}{mP_m}\int_{\sigma_0}^{0} \frac{\Phi'(\sigma)M_G^{-1}(\mu_F(\sigma))}{\Phi^2(\sigma)}d\sigma,$$

i.e., (12) holds, and through Lemma 3, F belongs to the convergence Φ-class with respect to G.

If $A < 0$ and $m = 1$, then $\mu_F(\sigma) = \mu_F(m\sigma)$, and (21) implies (12); i.e., through Lemma 3, the function F belongs to the convergence Φ-class with respect to the function G, Q.E.D.

Now, let F belong to the convergence Φ-class with respect to G, and F_1 is dominant. Then, (16) implies $\mu_{F_1}(\sigma) \leq (\mu_F(m\sigma)/c_1)^{1/m}$. Therefore, if $A \leq 0$, then $\mu_{F_1}(\sigma) \leq (\mu_F(\sigma)/c_1)^{1/m}$, and in view of the condition $M_G^{-1}(e^x) \in L^0$, we obtain $M_G^{-1}(\mu_{F_1}(\sigma) \leq c_2M_G^{-1}(\mu_F(\sigma)$, whence it follows that the function F_1 belongs to the convergence Φ-class with respect to the function G, provided that F belongs to the convergence Φ-class with respect to G.

The same conclusion can be made when $0 < A < +\infty$ and $m = 1$.

Finally, if $A = +\infty$ and $\Phi(m\sigma)/\Phi(\sigma) \leq p_m < +\infty$ for all $\sigma < +\infty$, then

$$\int_{\sigma_0}^{+\infty} \frac{\Phi'(\sigma)M_G^{-1}(\mu_{F_1}(\sigma))}{\Phi^2(\sigma)}d\sigma \leq \int_{\sigma_0}^{+\infty} \frac{\Phi'(\sigma)M_G^{-1}((\mu_F(m\sigma)/c_1)^{1/m})}{\Phi^2(\sigma)}d\sigma$$

$$\leq c_2\int_{\sigma_0}^{+\infty} \frac{\Phi'(\sigma)M_G^{-1}(\mu_F(m\sigma))}{\Phi^2(\sigma)}d\sigma \leq \frac{c_2p_m^2}{m}\int_{\sigma_0}^{+\infty} \frac{\Phi'(m\sigma)M_G^{-1}(\mu_F(m\sigma))}{\Phi^2(m\sigma)}d(m\sigma) < +\infty,$$

whence, by Lemma 3, it follows that the function F_1 belongs to the convergence Φ-class with respect to the function G. Since the function F_1 is dominant, all functions F_j belong to the convergence Φ-class with respect to the function G. The proof of Theorem 3 is thus completed. □

The following theorem indicates the conditions necessary for functions to belong to the convergence Φ-class with respect to $G \in S(\Lambda, 0)$.

Theorem 4. *Let $A \in (-\infty, +\infty]$, $F \in S(\Lambda, A)$, $\Phi \in \Omega(A)$, $\Phi(\sigma) = O(\Phi(\Psi(\sigma))$, as $\sigma \uparrow A$, and let the conditions (7) and (8) hold. Let $G \in S(\Lambda, 0)$, $(1/|M_G^{-1}(e^x)|) \in L^0$, and the function $F \in S(\Lambda, A)$ is the Hadamard composition of genus $m \geq 1$ of the function $F_j \in S(\Lambda, A)$, $1 \leq j \leq p$.*

If all F_j belong to the convergence Φ-class with respect to G and either $A > 0$ or $A = 0$ and $\Phi(\sigma)/\Phi(m\sigma) \leq P_m < +\infty$ for all $\sigma < 0$ or $A < 0$ and $m = 1$, then F belongs to the convergence Φ-class with respect to G.

If F belongs to the convergence Φ-class with respect to G, F_1 is dominant, and either $A \le 0$ or $A = +\infty$, and $\Phi(m\sigma)/\Phi(\sigma) \le p_m < +\infty$ for all $\sigma < +\infty$ or $0 < A < +\infty$ and $m = 1$; then, all F_j belong to the convergence Φ-class with respect to G.

Proof. As in proof Theorem 3, from (17) and (20), now, in view of the condition $(1/|M_G^{-1}(e^x)|) \in L^0$, we obtain

$$\frac{1}{|M_G^{-1}(\mu_F(m\sigma))|} \le \frac{1}{\left|M_G^{-1}\left(\sum\limits_{k_1+\cdots+k_p=m} |c_{k_1\ldots k_p}|\mu_{F_1}(\sigma)^{k_1}\cdot\ldots\cdot\mu_{F_p}(\sigma)^{k_p}\right)\right|} \le$$

$$\le K_2\left(\frac{1}{|M_G^{-1}(\mu_{F_1}(\sigma))|} + \cdots + \frac{1}{|M_G^{-1}(\mu_{F_1}(\sigma))|}\right),$$

i.e.,

$$\int\limits_{\sigma_0}^{A} \frac{\Phi'(\sigma)}{\Phi^2(\sigma)|M_G^{-1}(\mu_F(m\sigma))|}d\sigma \le$$

$$\le K_2\left(\int\limits_{\sigma_0}^{A} \frac{\Phi'(\sigma)}{\Phi^2(\sigma)|M_G^{-1}(\mu_{F_1}(\sigma))|}d\sigma + \cdots + \int\limits_{\sigma_0}^{A} \frac{\Phi'(\sigma)}{\Phi^2(\sigma)|M_G^{-1}(\mu_{F_p}(\sigma))|}d\sigma\right) < +\infty, \tag{22}$$

provided that all functions F_j belong to the convergence Φ-class with respect to the function $G \in S(\Lambda, 0)$.

If $A > 0$, then $\mu_F(\sigma) \le \mu_F(m\sigma)$ for $\sigma \in [0, A)$, and (22) implies (13); i.e., through Lemma 3, F belongs to the convergence Φ-class with respect to G.

If $A = 0$ and $\Phi(\sigma) \le P_m\Phi(m\sigma)$ for all $\sigma < 0$, then, as in the proof Theorem 3,

$$\int\limits_{\sigma_0}^{0} \frac{\Phi'(\sigma)}{\Phi^2(\sigma)|M_G^{-1}(\mu_F(\sigma))|}d\sigma \le mP_m^2 \int\limits_{\sigma_0}^{0} \frac{\Phi'(\sigma)}{\Phi^2(\sigma)|M_G^{-1}(\mu_F(m\sigma))|}d\sigma < +\infty,$$

i.e., (13) is true, and through Lemma 3, the function F belongs to the convergence Φ-class with respect to the function G.

If $A < 0$ and $m = 1$, then $\mu_F(\sigma) = \mu_F(m\sigma)$, and (22) implies (13); i.e., through Lemma 3, F belongs to the convergence Φ-class with respect to G, Q.E.D.

Now, let F belong to the convergence Φ-class with respect to G, and F_1 is dominant. If $A \le 0$, then , as above from the inequality $\mu_{F_1}(\sigma) \le (\mu_F(m\sigma)/c_1)^{1/m}$, in view of the condition $1/|M_G^{-1}(e^x)| \in L^0$, we obtain $1/|M_G^{-1}(\mu_{F_1}(\sigma)| \le c_2/|M_G^{-1}(\mu_F(\sigma)|$, whence it follows that F_1 belongs to the convergence Φ-class with respect to G, provided that F belongs to the convergence Φ-class with respect to G.

The same conclusion can be made when $0 < A < +\infty$ and $m = 1$.

Finally, if $A = +\infty$ and $\Phi(m\sigma)/\Phi(\sigma) \le p_m < +\infty$ for all $\sigma < +\infty$, then

$$\int\limits_{\sigma_0}^{+\infty} \frac{\Phi'(\sigma)}{\Phi^2(\sigma)|M_G^{-1}(\mu_F(\sigma))|}d\sigma \le \frac{c_2 p_m^2}{m} \int\limits_{\sigma_0}^{+\infty} \frac{\Phi'(m\sigma)}{\Phi^2(m\sigma)|M_G^{-1}(\mu_F(m\sigma))|}d(m\sigma) < +\infty,$$

i.e., F_1 belongs to the convergence Φ-class with respect to G. Since the function F_1 is dominant, all functions F_j belong to the convergence Φ-class with respect to the function G. The proof of Theorem 4 is thus completed. □

5. Examples

By choosing the functions G, F, and Φ in one way or another, we can obtain the corresponding statements from Theorems 3 and 4.

At first, let us assume that the entire Dirichlet series (2) reduces to an exponential monomial; i.e., $G(s) = g\exp\{s\lambda\}$. Then, $M_G(\sigma) = |g|\exp\{\sigma\lambda\}$ for all $\sigma \in (-\infty, +\infty)$ and $M_G^{-1}(x) = (\ln x - \ln|g|)/\lambda = (1+o(1))\ln x/\lambda$ as $x \to +\infty$. Therefore, if $\Phi \in \Omega(0)$, then the function $F \in S(\Lambda, 0)$ belongs to the convergence Φ-class with respect to G if, and only if,

$$\int_{\sigma_0}^{0} \frac{\Phi'(\sigma)\ln M_F(\sigma))}{\Phi^2(\sigma)} d\sigma < +\infty, \tag{23}$$

i.e., we arrive at the convergence Φ-class of the one considered in [14]. Let us choose again $\Phi(\sigma) = |\sigma|^{-(\eta+1)}$, where $\eta > 0$. Then, $\Phi \in \Omega(0)$. It is not difficult to establish the following properties:

$$\Phi(\sigma) = \Phi(m\sigma)m^{\eta+1} \text{ and } \frac{\Phi''(\sigma)\Phi(\sigma)}{(\Phi'(\sigma))^2} = \frac{\eta+2}{\eta+1}.$$

Choose $\Psi(\sigma) = -\frac{\eta+2}{\eta+1}|\sigma|$. Then, $\Phi(\Psi(\sigma)) = \left(\frac{\eta+1}{\eta+2}\right)^{\eta+1}\Phi(\sigma)$ and

$$\sum_{n=1}^{\infty} |f_n| \exp\left\{\lambda_n \Psi\left(\frac{1}{\lambda_n}\ln\frac{1}{|f_n|}\right)\right\} = \sum_{n=1}^{\infty} |f_n| \exp\left\{-\lambda_n \frac{\eta+2}{\eta+1}\left|\frac{1}{\lambda_n}\ln\frac{1}{|f_n|}\right|\right\} = \sum_{n=1}^{\infty} |f_n|^{-1/(\eta+1)}$$

because $|f_n| \to +\infty$ as $n \to \infty$; i.e., (8) holds if $\sum_{n=1}^{\infty} |f_n|^{-1/(\eta+1)} < +\infty$. With this choice of function Φ, relation (23) has the form $\int_{\sigma_0}^{0} |\sigma|^{\eta-1} \ln M_F(\sigma) d\sigma < +\infty$, and Theorem 3 implies the following statement.

Corollary 3. *Let the function $F \in S(\Lambda, 0)$ be the Hadamard composition of genus $m \geq 1$ of the functions $F_j \in S(\Lambda, 0)$, $1 \leq j \leq p$, let the function F_1 be dominant, and let and $\sum_{n=1}^{\infty} |f_n|^{-1/(\eta+1)} < +\infty$ for some $\eta > 0$. Then, in order that*

$$\int_{\sigma_0}^{0} |\sigma|^{\eta-1} \ln M_F(\sigma) d\sigma < +\infty,$$

it is necessary and sufficient that for all j,

$$\int_{\sigma_0}^{0} |\sigma|^{\eta-1} \ln M_{F_j}(\sigma) d\sigma < +\infty.$$

Now, let

$$G(s) = \exp\{e^{\varrho s}\} = \sum_{n=0}^{\infty} \frac{e^{\varrho sn}}{n!}, \quad 0 < \varrho < +\infty.$$

Then, $G \in S(\mathbb{Z}_+, +\infty)$, $M_G(\sigma) = \exp\{e^{\varrho\sigma}\}$, and $M_G^{-1}(x) = (\ln\ln x)/\varrho$ for $x > e$. Therefore, if $\Phi \in \Omega(+\infty)$, then the function $F \in S(\mathbb{Z}_+, +\infty)$ belongs to the convergence Φ-class with respect to G if, and only if,

$$\int_{\sigma_0}^{\infty} \frac{\Phi'(\sigma)\ln\ln M_F(\sigma))}{\Phi^2(\sigma)} d\sigma < +\infty.$$

We choose $\Phi \in \Omega(+\infty)$ such that $\Phi(\sigma) = \sigma^{1+\eta}$ for $\sigma \geq 1$, where $\eta > 0$. Then, it is easy to check that $\frac{\Phi''(\sigma)\Phi(\sigma)}{(\Phi'(\sigma))^2} = \frac{\eta}{1+\eta}$. Put $\Psi(\sigma) = \frac{\eta}{1+\eta}\sigma$ and calculate $\Phi(\Psi(\sigma)) = \left(\frac{\eta}{1+\eta}\right)^{1+\eta}\Phi(\sigma)$, $\Phi(m\sigma) = \Phi(\sigma)m^{\eta+1}$ and

$$\sum_{n=1}^{\infty}|f_n|\exp\left\{\lambda_n\Psi\left(\frac{1}{\lambda_n}\ln\frac{1}{|f_n|}\right)\right\} = \sum_{n=1}^{\infty}|f_n|\exp\left\{\frac{\eta}{1+\eta}\ln\frac{1}{|f_n|}\right\} = \sum_{n=1}^{\infty}|f_n|^{1/(\eta+1)}.$$

Therefore, Theorem 3 implies the following statement.

Corollary 4. *Let the function $F \in S(\mathbb{Z}_+, +\infty)$ be the Hadamard composition of genus $m \geq 1$ of the functions $F_j \in S(\mathbb{Z}_+, +\infty)$, $1 \leq j \leq p$, let the the function F_1 be dominant, and let $\sum\limits_{n=1}^{\infty}|f_n|^{1/(\eta+1)} < +\infty$ for some $\eta > 0$. Then, in order that*

$$\int_{\sigma_0}^{\infty}\frac{\ln\ln M_F(\sigma)}{\sigma^{p+2}}d\sigma < +\infty,$$

it is necessary and sufficient that for all j,

$$\int_{\sigma_0}^{\infty}\frac{\ln\ln M_{F_j}(\sigma)}{\sigma^{p+2}}d\sigma < +\infty.$$

Finally, if $G(s) = e^{-1/s}$, then, for $\sigma < 0$, we have

$$|G(\sigma+it)| = \exp\left\{\mathrm{Re}\frac{-1}{\sigma+it}\right\} = \exp\left\{\mathrm{Re}\frac{-\sigma-it}{\sigma^2+t^2}\right\} = \exp\left\{\frac{|\sigma|}{|\sigma|^2+t^2}\right\},$$

i.e., $M_G(\sigma) = e^{1/|\sigma|}$, $\exp\left\{\frac{1}{M_G^{-1}(x)}\right\} = x$; thus, $\frac{1}{|M_G^{-1}(M_F(\sigma))|} = \ln M_F(\sigma)$. Therefore, (11) holds with $A = +\infty$ if $\int\limits_{\sigma_0}^{+\infty}\frac{\Phi'(\sigma)\ln M_F(\sigma))}{\Phi^2(\sigma)}d\sigma < +\infty$. We choose $\Phi(\sigma) = e^{\varrho\sigma}$, $0 < \varrho < +\infty$. Then, $\frac{\Phi''(\sigma)\Phi(\sigma)}{(\Phi'(\sigma))^2} = \varrho$, $\Psi(\sigma) = \sigma - 1/\varrho$, $\Phi(\Psi(\sigma)) = \Phi(\sigma)/e$ and

$$\sum_{n=1}^{\infty}|f_n|\exp\left\{\lambda_n\Psi\left(\frac{1}{\lambda_n}\ln\frac{1}{|f_n|}\right)\right\} = \sum_{n=1}^{\infty}|f_n|\exp\left\{\ln\frac{1}{|f_n|} - \frac{\lambda_n}{\varrho}\right\} = \sum_{n=1}^{\infty}\exp\left\{-\frac{\lambda_n}{\varrho}\right\} < +\infty$$

provided $\ln n = o(\lambda_n)$ as $n \to \infty$. Therefore, Theorem 4 implies the following statement.

Corollary 5. *Let the function $F \in S(\Lambda, +\infty)$ be the Hadamard composition of the genus $m \geq 1$ of the functions $F_j \in S(\Lambda + \infty)$ and $\ln n = o(\lambda_n)$, as $n \to \infty$. If $\int\limits_{\sigma_0}^{+\infty} e^{-\varrho\sigma}\ln M_{F_j}(\sigma)d\sigma < +\infty$ for all j, then $\int\limits_{\sigma_0}^{+\infty} e^{-\varrho\sigma}\ln M_F(\sigma)d\sigma < +\infty$.*

Note that for entire function f of order ϱ G. Valiron ([22], p. 18) introduced the convergence class via the condition $\int\limits_{1}^{\infty}\frac{\ln M_f(r)}{r^{\varrho+1}}dr < +\infty$, where $M_f(r) = \max\{|f(z)| : |z| = r\}$;

and P.K. Kamthan [23] extended the concept of the Valiron class to the entire Dirichlet series, defining the convergence class by the condition $\int\limits_{\sigma_0}^{+\infty} e^{-\varrho\sigma} \ln\, M_F(\sigma) d\sigma < +\infty$.

6. Discussion

In view of Theorem 3. the following question arises:

Problem 1. *What is a connection between the functions F_j belonging to the convergence Φ-class with respect to $G \in S(\Lambda, +\infty)$ and those belonging to this class of their Hadamard composition F in the following two non-overlapping cases:*

I. $A \le 0$ and $m \ge 2$ in the first part of Theorem 3;

II. $0 < A < +\infty$ and $m \ge 2$ in the second part of Theorem 3.

Moreover, Theorem 4 generates the same situation:

Problem 2. *What is a connection between the functions F_j belonging to the convergence Φ-class with respect to $G \in S(\Lambda, +\infty)$ and those belonging to this class of their Hadamard composition F in the following two cases:*

I. $A \le 0$ and $m \ge 2$ in the first part of Theorem 4;

II. $0 < A < +\infty$ and $m \ge 2$ in the second part of Theorem 4.

At the present moment, we are not ready to give a full answer to the above questions.

7. Conclusions

Theorems 3 and 4 represent very general and technical results. But they admit many partial cases as corollaries for different choices of the functions G, F, and Φ (see Section 5). This is the primary significance of the obtained results. There are many directions for further generalizations: multiple Dirichlet series, Taylor–Dirichlet-type series, hyper-Dirichlet series, and so on.

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