

# Application of group-theoretical methods to solving the point explosion problem in incompressible liquid



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## ABSTRACT

Group-theoretical methods are applied to find particular solutions to differential systems in hydrodynamics. Obtained solutions that satisfy the Rankine–Hugoniot conditions is used to describe a point explosion in water.

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## 1. Introduction

There is no general theory of explosion effects in a medium now. Taking into consideration all the factors that affect a shock wave propagation in the real world makes the issue of theoretical description of this phenomenon a rather complicated problem. Its successful solving needs an idealization, specifically taking into account only the factors that are prevalent in description of the phenomenon. As an example of such idealization we can suppose that the energy release is immediate, and the volume that an explosive occupies and the mass of an explosive charge are zero [1]. This means that in the case of spherical symmetry an explosion will take place in a point, in the case of cylindrical symmetry it will be along a straight line, and for plane symmetry in a plane. The term “point explosion” denotes also explosions along a line and in a plane, and these practically corresponds to those of solitary cylindrical charges or charges placed in a plane, respectively.

In spite of the idealization accepted in the formulation of the point explosion problem, the point explosion theory enables one to obtain, with reasonable accuracy for practical purposes, all necessary data on the patterns of unsteady motion of the medium produced by explosions. This is due to the fact that in natural processes of explosion considerable energy emissions always last for quite short periods of time, and take place in a small volume compared with the volume occupied by the environment.

In the present paper we consider the problem of point explosion in water. The results obtained enhance the study [1], which was conducted for the case of ideal gas.

## 2. Definitions and formulation of the problem

Nonviscous liquid motion can be described by a system of first-order quasilinear differential equations. The first two equations which represent Newton's second law and the mass conservation law do not depend on the nature of the process and have the form

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$$\begin{aligned} \frac{\partial \vec{v}_k}{\partial t} + \vec{v}_i \frac{\partial \vec{v}_k}{\partial x_i} + \frac{1}{\rho} \frac{\partial p}{\partial x_k} &= 0, \\ \frac{\partial \rho}{\partial t} + \vec{v}_k \frac{\partial \rho}{\partial x_k} + \rho \frac{\partial}{\partial x_k} \vec{v}_k &= 0, \end{aligned} \quad (1)$$

where  $\vec{v}_k = \vec{v}_k(t, x_1, x_2, x_3)$  is the  $k$ -th velocity component in a point  $\vec{x} = (x_1, x_2, x_3)$  at the time  $t$ ,  $p$  is the pressure and  $\rho$  is the liquid density.

For complete description of liquid motion we adduce to the system (1) a state equation, which establishes a dependence among the pressure, the density and the temperature, as well as the equations that reflect the features of the process in question (e.g., the entropy conservation, the constant value of the temperature etc.).

There is no generally accepted state equation for liquid, unlike for ideal gas [1]. Known state equations are applicable in relatively narrow ranges of values of thermodynamical parameters. In many cases for liquid it may be useful to consider simplified state equations, such that the corresponding system of the equations of motion can be in some way integrated. The simplest example is the model of incompressible fluid,

$$\rho = \rho_1 = \text{const.}$$

For this case it is possible to obtain an exact solution of the one-dimensional problem.

Dynamical and statical compressibility of water in a wide range of pressure values can be described satisfactorily by the state equation in the Tate form:

$$p = B(T) [(\rho/\rho_0)^\kappa - 1], \quad (2)$$

where  $B(T)$  is a function of the temperature  $T$ , and the value of the parameter  $\kappa$  for water is close to 7.

If a system of differential equations is invariant with respect to a group of continuous transformations, then the group-theoretical methods can be applied to find particular solutions. A large number of publications are devoted to group methods and their applications to differential equations (see [2–8] and references therein). The framework named *group analysis of differential equations*, as well as its numerous applications, is described in detail in [2].

The basic definitions used in the present paper are as follows. Let

$$F^N(t, \vec{x}, \vec{v}, \rho, p, T, D\vec{v}, \dots, D^N \vec{v}, \dots, D^N T) = 0 \quad (3)$$

be a system of differential equations, where  $\vec{x} = (x_1, x_2, x_3)$ ,  $\vec{v} = (v_1, v_2, v_3)$ , and  $D^k$  is the  $k$ -th order derivative with respect to  $x$ .

**Definition 1.** System (3) is said to admit a group  $G^r$  of continuous transformations acting in the space of variables  $\sigma = (t, \vec{x}, \vec{v}, \rho, p, T) \in \mathbb{R}^{10}$ , if for any transformation  $\sigma' = \varphi(\sigma)$ ,  $\varphi \in G^r$ , the system remains unchanged.

Let  $\Psi$  be a manifold in the space  $\mathbb{R}^m$  defined by the system of equations

$$\Psi^k(\sigma) = 0, \quad \sigma \in \mathbb{R}^m, \quad k = 1, 2, \dots, n, \quad (4)$$

and  $G^r$  a group of continuous transformations acting in  $\mathbb{R}^m$ .

**Definition 2.**  $\Psi$  is called an *invariant manifold* of the group  $G^r$  if for each  $\sigma \in \Psi$  and  $\varphi \in G^r$  the point  $\sigma' = \varphi(\sigma)$  belongs to  $\Psi$ .

A manifold  $\Psi$  is called *regularly defined* if the rank of the Jacobi matrix  $(\partial \Psi^j / \partial x^\mu)$  on the space  $\mathbb{R}^m$  equals the rank of this matrix on the set  $\Psi$ . For regularly defined manifolds the following theorem holds.

**Theorem 1.** Let  $L_r$  be the Lie algebra of the group  $G^r$ . A manifold  $\Psi$  regularly defined by Eq. (4) is an invariant manifold of the group  $G^r$  if and only if for any operator  $\xi \cdot \partial \in L_r$

$$(\xi \cdot \partial) \Psi^k(\sigma)|_\Psi = 0, \quad k = 1, 2, \dots, n.$$

Let  $H$  be a subgroup of the group  $G^r$  admitted by system (3).

**Definition 3.** A solution  $W = (\vec{v}, \rho, p, T)$  is said to be *invariant with respect to  $H$* , or  *$H$ -invariant*, if  $W$  considered as a subset of points of the space  $\mathbb{R}^{10}$  is an invariant manifold of the group  $H$ .

To describe the invariants of the group  $H^i \subset G^r$  it is sufficient to find the solutions of the equation

$$(\xi_j \cdot \partial) J = 0, \quad j = 1, 2, \dots, i, \quad (5)$$

where  $\{\xi_j \cdot \partial\}_{j=1}^i$  is a basis of the Lie algebra of the group  $H^i \subset G^r$ . Any invariant of the group  $H^i$  can be determined as a function of a finite number of independent solutions  $J_1, J_2, \dots, J_i$  of system (5) which are called the *functional basis of invariants* [2].

To obtain  $H^i$ -invariant solutions of the system (1) we present the solutions  $W = (t, \vec{x}, \dots, T)$  as functions of  $J_1, J_2, \dots, J_i$ . Thus the system in question transforms to a so-called factor-system that connects only the invariants of the group  $H^i$ . This approach allows to reduce the number of independent variables and thereby to simplify the problem.

In what follows we consider solutions of system (1) that are invariant with respect to some subgroups of the Euclidian group  $E^3$ , namely:

- 1) solutions that are invariant with respect to any shifts in the plane  $\mathbb{R}^2(x_2, x_3)$  (plane waves);
- 2) solutions that are invariant with respect to shifts along the axis  $x_3$  and the rotations around the axis  $x_3$  (cylindrical waves);
- 3) solutions that are invariant with respect to the complete group of rotations in  $\mathbb{R}^3$  (spherical waves).

The factor-systems for all these cases can be presented in a uniform manner,

$$\frac{\partial}{\partial t} v + v \frac{\partial}{\partial r} v + \frac{1}{\rho} \frac{\partial}{\partial r} p = 0, \quad \frac{\partial}{\partial t} \rho + v \frac{\partial}{\partial r} \rho + \rho \left( \frac{\partial v}{\partial r} + v \frac{v}{r} \right) = 0, \tag{6}$$

where

$$\begin{aligned} v = 0, \quad r = x_1, \quad v = v_1 \quad &\text{for planar waves,} \\ v = 1, \quad r = \sqrt{x_1^2 + x_2^2}, \quad v = \sqrt{v_1^2 + v_2^2} \quad &\text{for cylindrical waves,} \\ v = 2, \quad r = \sqrt{x_1^2 + x_2^2 + x_3^2}, \quad v = \sqrt{v_1^2 + v_2^2 + v_3^2} \quad &\text{for spherical waves.} \end{aligned}$$

In the present paper we look for solutions of system (6) that are invariant with respect to the group of scale transformations. Such solutions are said to be *self-similar*, or *automodel*. In Section 3 we find self-similar solutions of the factor-system (6) under the assumption  $\rho = \rho_1 = \text{const}$ . In Section 4 we look for self-similar solutions to system (6) with the state equation of form (2) with  $B = \text{const}$  (the isothermal case). In Section 5 solutions of (6) with the state equation of the form  $\rho = \phi(T)\theta(\rho/\rho_0)$  provided that

$$\frac{\partial T}{\partial r} = 0 \tag{7}$$

are considered. We also determine the functions  $\phi$  and  $\theta$  such that the system of Eqs. (6) and (7) admits the group of scale transformations. For the state equation in the Tate form a self-similar solution of the point explosion problem is found that satisfies the Rankine–Hugoniot conditions at the shock wave front.

### 3. Exact solutions of explosion problem in incompressible liquid

Under the constraint  $\rho = \rho_1 = \text{const}$  system (6) is equivalent to the following system of differential equations:

$$\frac{\partial v}{\partial t} + v \frac{\partial v}{\partial r} + \frac{1}{\rho_1} \frac{\partial p}{\partial r} = 0, \quad \frac{\partial v}{\partial r} + v \frac{v}{r} = 0, \quad v = 0, 1, 2. \tag{8}$$

It can be verified by direct calculation that Eqs. (8) admit the group of scale transformations of the form

$$\begin{pmatrix} t \\ r \\ v \\ p \end{pmatrix} \rightarrow \begin{pmatrix} t' \\ r' \\ v' \\ p' \end{pmatrix} = \begin{pmatrix} e^{\alpha t} \\ e^{\beta r} \\ e^{\beta-\alpha} v \\ e^{2(\beta-\alpha)} p \end{pmatrix}, \tag{9}$$

where  $\alpha$  and  $\beta$  are parameters of the transformations. It is easy to see that basis elements of the Lie algebra of group (9) have the form

$$Z_1 = t \frac{\partial}{\partial t} - v \frac{\partial}{\partial v} - 2p \frac{\partial}{\partial p}, \quad Z_2 = r \frac{\partial}{\partial r} + v \frac{\partial}{\partial v} + 2p \frac{\partial}{\partial p}.$$

Thus the invariants of the transformation group (9) satisfy the equation

$$(\alpha_1 Z_1 + \alpha_2 Z_2)J = 0, \tag{10}$$

where  $\alpha_1$  and  $\alpha_2$  are arbitrary real parameters.

Eq. (10) has the following independent solutions:

$$\frac{r}{t^\delta} = c_0, \quad \frac{v}{t^{\delta-1}} = c_1, \quad \frac{p}{t^{2(\delta-1)}} = c_2, \tag{11}$$

where  $\delta = \alpha_2/\alpha_1$ .

An arbitrary invariant of group (9) can be expressed by the functional relation

$$\phi\left(\frac{r}{t^\delta}, \frac{v}{t^{\delta-1}}, \frac{p}{t^{2(\delta-1)}}\right) = 0,$$

which connects the independent invariants (11). We seek self-similar solutions of system (8) in form of functions

$$v(t, r) = t^{\delta-1} \varphi(\xi), \quad p(t, r) = t^{2(\delta-1)} \psi(\xi), \quad \xi = \frac{r}{t^\delta}. \quad (12)$$

Substituting (12) into (8) we represent the equations for incompressible liquid in the form of a system of ordinary differential equations,

$$(\varphi - \delta\xi) \frac{d\varphi}{d\xi} + \frac{1}{\rho_1} \frac{d\psi}{d\xi} + (\delta - 1)\varphi = 0, \quad \xi \frac{d\varphi}{d\xi} + v\varphi = 0. \quad (13)$$

The solution of system (13) has the form

$$\begin{aligned} \psi &= (1 - \delta)c_1\xi + c_2, & \varphi &= c_1, & v &= 0, \\ \psi &= \rho_1 \left[ (1 - 2\delta)c_2 \log \xi - \frac{c_1^2}{2\xi^2} \right] + c_2, & \varphi &= \frac{c_1}{\xi}, & v &= 1, \\ \psi &= \rho_1 \left[ (3\delta - 1)c_1 - \frac{c_1^2}{2\xi^4} \right] + c_2, & \varphi &= \frac{c_1}{\xi^2}, & v &= 2, \end{aligned} \quad (14)$$

where  $c_1$  and  $c_2$  are arbitrary constants.

Consider more closely the solution that corresponds to the spherical symmetry. Suppose that a point explosion occurred at the moment  $t = 0$  in an infinite volume of immobile liquid, and as a result the energy  $E_0$  released. As the liquid is suggested to be incompressible without heat conductivity, all the ejected energy becomes the kinetic energy of the liquid,

$$E_0 = 4\pi \int_{r_*}^{\infty} \frac{\rho v^2}{2} r^2 dr = 2\pi\rho_1 c_1^2 \frac{t^{2(3\delta-1)}}{r_*(t)}, \quad (15)$$

where  $r_*$  is the radius of the cavern formed by the explosion. The function  $r_*$  meets the equation

$$\frac{dr_*}{dt} = v_*(t) = \frac{c_1}{(r/t^\delta)^2} t^{\delta-1} \Big|_{r=r_*} = \frac{c_1 t^{3\delta-1}}{r_*^2}. \quad (16)$$

Hence

$$r_*(t) = \left( \frac{c_1}{3\delta} \right)^{1/3} t^\delta,$$

and from (15) we deduce

$$E_0 = 2\pi\rho_1 t^{5\delta-2} (3\delta)^{1/3} c_1^{5/3}.$$

As  $E_0$  does not depend on time, it is necessary to assume  $\delta = 2/5$ . Now we can express the constant  $c_1$  in terms of the explosion energy and the density:

$$c_1 = \left[ \frac{E_0}{2\pi\rho_1 (6/5)^{1/3}} \right]^{3/5}. \quad (17)$$

Taking into account the boundary conditions at infinity,  $p(t, x \rightarrow \infty) = 0$ , we deduce that the self-similar solution to the explosion problem in incompressible liquid has the form

$$v = \left[ \frac{E}{2\pi\rho(6/5)^{1/3}} \right]^{3/5} \frac{t^{1/5}}{r^2}, \quad \rho = t^{-4/5} \frac{\rho_1 \left[ \frac{E}{2\pi\rho_1(6/5)^{1/3}} \right]^{3/5}}{2r^4} \frac{6}{5} [r^3 - r_*^3]. \quad (18)$$

**Remark 1.** In accordance to the physical meaning of the problem, Eqs. (18) hold for  $r \geq r_*$ . If  $0 \leq r \leq r_*$  then the value of  $\rho$  should be set equal to zero. The solution (18) was first obtained by L.I. Sedov.

#### 4. Isothermal self-similar solutions

Consider the case when the state equation is expressed by (2) and the function  $B$  does not depend on the temperature, i.e. the process is isothermal. Substituting (2) into (6) we obtain the following system of equations:

$$\frac{\partial v}{\partial t} + v \frac{\partial v}{\partial r} + M\rho^{\alpha-2} \frac{\partial p}{\partial r} = 0, \quad \frac{\partial \rho}{\partial t} + v \frac{\partial \rho}{\partial r} + \rho \left( \frac{\partial v}{\partial r} + v \frac{v}{r} \right) = 0, \quad (19)$$

where  $M = \frac{\alpha B}{\rho_0}$ . The system (19) admits the group of scale transformations

$$\begin{pmatrix} t \\ r \\ v \\ p \end{pmatrix} \rightarrow \begin{pmatrix} t' \\ r' \\ v' \\ p' \end{pmatrix} = \begin{pmatrix} e^{\alpha} t \\ e^{\beta} r \\ e^{\beta-\alpha} v \\ e^{2(\beta-\alpha)/(\alpha-1)} p \end{pmatrix}, \tag{20}$$

The arbitrary element of the Lie algebra of the group (20) is of the form

$$Z = \alpha_1 t \frac{\partial}{\partial t} + \alpha_2 r \frac{\partial}{\partial r} + (\alpha_2 - \alpha_1) v \frac{\partial}{\partial v} + 2 \frac{\alpha_2 - \alpha_1}{\alpha - 1} \rho \frac{\partial}{\partial \rho}.$$

Solving the equation  $ZJ = 0$  we have a basis of the invariants for group (20),

$$\frac{r}{t^\delta} = c_1, \quad \frac{v}{t^{\delta-1}} = c_2, \quad \frac{p}{t^{2(\delta-1)/(\alpha-1)}} = c_3, \tag{21}$$

where  $\delta = \alpha_2/\alpha_1$ . Thereafter, we expect that the self-similar solutions of system (19) have the form

$$v = t^{\delta-1} \varphi(\xi), \quad \rho = t^{2\frac{\delta-1}{\alpha-1}} \psi(\xi), \quad \xi = \frac{r}{t^\delta}. \tag{22}$$

We substitute (22) into (19) and obtain a system of ordinary differential equations,

$$\begin{aligned} (\varphi - \delta\xi) \frac{d\varphi}{d\xi} + M\psi^{\alpha-2} \frac{d\psi}{d\xi} + (\delta - 1)\varphi &= 0, \\ \psi \frac{d\varphi}{d\xi} + (\varphi - \delta\xi) \frac{d\psi}{d\xi} + \psi \left( v \frac{\varphi}{\xi} + 2 \frac{\delta - 1}{\alpha - 1} \right) &= 0. \end{aligned} \tag{23}$$

A particular solution to system (23) can be obtained if we set  $\varphi = \delta\xi$ . The second equation of (23) yields the following constraint for the parameter  $\delta$ :

$$\delta = \frac{2}{2 + (v + 1)(\alpha - 1)}. \tag{24}$$

Substituting the value of the function  $\varphi$  into the first Eq. (23) we have

$$M\psi^{\alpha-2} \frac{d\psi}{d\xi} = (1 - \delta)\delta\xi.$$

Solving this equation we find a particular solution to system (23),

$$\varphi = \delta\xi, \quad \psi = \left[ c_1 + \frac{\alpha - 1}{M} \delta(1 - \delta) \frac{\xi^2}{2} \right]^{\frac{1}{\alpha-1}}. \tag{25}$$

We assume that  $\alpha \neq 1$  in (25).

After setting  $\delta = 1/2$  we can rewrite the first equation of system (23) in the form of a derivative,

$$\frac{d}{d\xi} \left[ \frac{M}{\alpha - 1} \psi^{\alpha-1} + \frac{\varphi^2}{2} - \delta\xi\varphi \right] = 0. \tag{26}$$

Under the restriction  $\alpha \neq 1$  we obtain the following relationship between the functions  $\psi$  and  $\varphi$ :

$$\psi = \left\{ \frac{\alpha - 1}{M} \left[ \left( \delta\xi - \frac{\varphi}{2} \right) \varphi + c_3 \right] \right\}^{\frac{1}{\alpha-1}}. \tag{27}$$

Substituting (27) into the second equation of (23), and then applying the change of variables

$$\varphi(\xi) = \xi x \tag{28}$$

we have the equation

$$\frac{\xi}{x} \frac{dx}{d\xi} = \frac{(\frac{1}{\alpha-1} - vx) [\xi^2(1-x)x + c_3] - \xi^2 \frac{x(x-1/2)}{\alpha-1}}{x [\xi^2 x(1-x) + c_3 + \xi^2 \frac{x-1/2}{\alpha-1} (1-2x)]} - 1. \tag{29}$$

Under the restriction  $c_3 = 0$  it is possible to separate the variables in (29):

$$\frac{d\xi}{\xi} = dx \frac{2(\alpha - 1)x(1 - x) - [1 - (2x)^2]}{2[1 - (\alpha - 1)vx](1 - x) - (2x - 1)[2(\alpha - 1)x(1 - x) - (1 - (2x)^2)]}. \tag{30}$$

Eq. (30) may be integrated for any values of  $\alpha$  and  $v$  ( $v = 0, 1, 2$ ).

In the case when  $\delta$  is an arbitrary constant, system (23) has no closed-form solutions. However, this system can be transformed to a first-order ordinary differential equation.

Using algebraic manipulations we reduce system (23) to the following equivalent form:

$$\begin{aligned}\xi \left[ M\psi^{\alpha-1} - (\varphi - \delta\xi)^2 \right] \frac{d\psi}{d\xi} &= \psi(\varphi - \delta\xi) \left( v\varphi + 2\frac{\delta-1}{\alpha-1}\xi \right) - (\delta-1)\xi\varphi\psi, \\ \xi \left[ M\psi^{\alpha-1} - (\varphi - \delta\xi)^2 \right] \frac{d\varphi}{d\xi} &= \xi(\delta-1)(\varphi - \delta\xi)\varphi - M\psi^{\alpha-1} \left( v\varphi + 2\frac{\delta-1}{\alpha-1}\xi \right).\end{aligned}\quad (31)$$

Suppose that  $\xi \left[ M\psi^{\alpha-1} - (\varphi - \delta\xi)^2 \right] \neq 0$  and apply the substitution  $\psi^{\alpha-1} = \xi^\mu y$ ,  $\varphi = \xi^\alpha x$ . System (23) herewith transforms to the following equivalent system of equations:

$$\begin{aligned}\xi^\mu \frac{dy}{d\xi} &= \frac{y^{\xi^\mu} \left\{ \alpha \left[ (\xi^\alpha x - \delta\xi) (v\xi^\alpha x + 2\frac{\delta-1}{\alpha-1}\xi) - (\delta-1)\xi^{\alpha+1} x \right] - \mu \left[ M\xi^\mu y - (\xi^\alpha x - \delta\xi)^2 \right] \right\}}{\xi \left[ M\xi^\mu y - (\xi^\alpha x - \delta\xi)^2 \right]}, \\ \frac{dx}{d\xi} &= \frac{\xi(\delta-1)(\xi^\alpha x - \delta\xi)\xi^\alpha x - M\xi^\mu y (v\xi^\alpha x + 2\frac{\delta-1}{\alpha-1}\xi) - \alpha\xi^\alpha x \left[ M\xi^\mu y - (\xi^\alpha x - \delta\xi)^2 \right]}{\xi^{\alpha+1} \left[ M\xi^\mu y - (\xi^\alpha x - \delta\xi)^2 \right]}.\end{aligned}\quad (32)$$

If  $\alpha = 1$  and  $\mu = 2$  then the numerators of the right hand sides of both Eqs. (32) are the homogeneous functions of power three with respect to  $\xi$ . Therefore, we divide the first equation by the second and obtain a first-order ordinary differential equation,

$$\frac{dy}{dx} = \frac{y \left\{ \alpha \left[ (x - \delta) (vx + 2\frac{\delta-1}{\alpha-1}) - (\delta-1)x \right] - 2 \left[ My - (x - \delta)^2 \right] \right\}}{(\delta-1)(x - \delta)x - My(vx + 2\frac{\delta-1}{\alpha-1}) - x \left[ My - (x - \delta)^2 \right]}.\quad (33)$$

Eq. (33) can be solved using computer.

## 5. Homothermal self-similar solution of explosion problem in compressible liquid

Supposing that there is a strong heat exchange in the medium, the problem may be solved within the following formulation.

$$\begin{aligned}\frac{\partial v}{\partial t} + v \frac{\partial v}{\partial r} + \frac{1}{\rho} \frac{\partial p}{\partial r} &= 0, \\ \frac{\partial \rho}{\partial t} + v \frac{\partial \rho}{\partial r} + \rho \left( \frac{\partial v}{\partial r} + v \frac{v}{r} \right) &= 0, \\ \frac{\partial T}{\partial r} &= 0, \\ p &= \phi(T)\theta(\rho/\rho_0).\end{aligned}\quad (34)$$

Expressing  $p$  in the first equation in terms of  $\rho$  and  $T$  we obtain the differential system

$$\begin{aligned}\frac{\partial v}{\partial t} + v \frac{\partial v}{\partial r} + \frac{\phi(T)}{\rho\rho_0} \eta(\rho/\rho_0) \frac{\partial \rho}{\partial r} &= 0, \\ \frac{\partial \rho}{\partial t} + v \frac{\partial \rho}{\partial r} + \rho \left( \frac{\partial v}{\partial r} + v \frac{v}{r} \right) &= 0, \\ \frac{\partial T}{\partial r} &= 0,\end{aligned}\quad (35)$$

where  $\eta(z) = d\theta(z)/dz$ .

The general form of the scale transformation of the independent and dependent variables involved in system (35) is

$$\begin{pmatrix} t \\ r \\ v \\ p \\ T \end{pmatrix} \rightarrow \begin{pmatrix} t' \\ r' \\ v' \\ p' \\ T' \end{pmatrix} = \begin{pmatrix} e^{\alpha t} \\ e^{\beta r} \\ e^{\gamma v} \\ e^{\epsilon p} \\ e^{\delta T} \end{pmatrix}.\quad (36)$$

From the second equation of (35) we deduce that the parameters  $\alpha, \beta$  and  $\gamma$  in admissible transformations for system (35) necessarily meet the condition

$$\gamma = \beta - \alpha.\quad (37)$$

Restrictions on the parameters  $\tau$  and  $\lambda$  will be obtained when we consider the first equation of system (35). These restrictions depend on the specific forms of the functions  $\phi(z)$  and  $\theta(z)$ .

We carry out the classification of the functions  $\phi$  and  $\theta$  that allow a representation of system (35) in automodel variables, considering the following cases:

- I.  $\lambda = 0, \tau = 0,$
- II.  $\lambda \neq 0, \tau = 0,$
- III.  $\lambda = 0, \tau \neq 0,$
- IV.  $\lambda \neq 0, \tau \neq 0.$

**Case I.**  $\phi(z)$  and  $\theta(z)$  are arbitrary functions,  $\gamma = 0$ . Therefore, by virtue of Eq. (37),  $\alpha = \beta$ . The Lie algebra is one-dimensional in this case and it is generated by the operator

$$Z = t \frac{\partial}{\partial t} + r \frac{\partial}{\partial r}. \tag{38}$$

An arbitrary invariant can be expressed as a function  $W(r/t) = 0$ , and the transition to automodel variables can be done by the substitution

$$v = \tilde{v}(\xi), \quad \rho = \rho_0 R(\xi), \quad T = \theta(\xi), \quad \xi = \frac{r}{t}. \tag{39}$$

The last equation of system (35) implies that the only possible self-similar solution for  $T$  is the solution  $\theta(\xi) = \text{const} = \Lambda$ .

After simple algebraic manipulations we obtain a system of ordinary differential equations,

$$(\tilde{v} - \xi) \frac{d\tilde{v}}{d\xi} + L \frac{\psi(R)}{R} \frac{dR}{d\xi} = 0, \quad R \frac{d\tilde{v}}{d\xi} + (\tilde{v} - \xi) \frac{dR}{d\xi} + v \frac{\tilde{v}R}{\xi} = 0, \tag{40}$$

where  $L = \phi(T)/\rho_0 \equiv \phi(\Lambda)/\rho_0$  and  $\psi(R) = \frac{d}{dR} \theta(R)$ .

**Case II.** Under these conditions for existence of self-similar solutions it is necessary to demand  $\phi(z) \sim z^\mu$ , where  $\mu$  is a real parameter. The function  $\theta(R)$  may be arbitrary.

From the first equation of system (35) we have the restriction on  $\lambda$ ,

$$\lambda = 2 \frac{\beta - \alpha}{\mu}. \tag{41}$$

The equation for invariants is

$$\left\{ \alpha_1 t \frac{\partial}{\partial t} + \alpha_2 r \frac{\partial}{\partial r} + (\alpha_2 - \alpha_1) v \frac{\partial}{\partial v} + \frac{2}{\mu} (\alpha_2 - \alpha_1) T \frac{\partial}{\partial T} \right\} J = 0.$$

An arbitrary solution of this equation can be presented in the form

$$W\left(\frac{r}{t^\delta}, \frac{v}{t^{\delta-1}}, \frac{T}{t^{2(\delta-1)/\mu}}\right) = 0,$$

where  $W$  is an arbitrary function.

In accordance with this, we transfer to equations with the invariant variables using the substitution

$$v = t^{\delta-1} \tilde{v}(\xi), \quad \rho = \rho_0 R(\xi), \quad T = t^{2(\delta-1)/\mu} \theta(\xi), \quad \xi = \frac{r}{t^\delta}, \tag{42}$$

where  $\delta = \alpha_2/\alpha_1$ . Since the variable  $T$  do not depend on  $r$ , we set  $\theta(\xi) = c_1 = \text{const}$ , and the system transforms to the following system of ordinary differential equations:

$$(\tilde{v} - \delta\xi) \frac{d\tilde{v}}{d\xi} + \Lambda \frac{\psi(R)}{R} \frac{dR}{d\xi} + (\delta - 1) \tilde{v} = 0, \quad R \frac{d\tilde{v}}{d\xi} + (\tilde{v} - \delta\xi) \frac{dR}{d\xi} + v \frac{\tilde{v}R}{\xi} = 0, \tag{43}$$

where  $\psi(R) = d\theta/dR$ ,  $\Lambda = \text{const}$ .

**Case III.** The function  $\phi(z)$  is arbitrary,  $\theta$  meets the condition  $d\theta/dR = LR^{\alpha-1}$ , where  $\alpha$  is a real parameter (we suggest that  $\alpha \neq 1$ ). The restriction on  $\tau$  is

$$\tau = \frac{2(\delta - 1)}{\alpha - 1}, \tag{44}$$

where  $\delta = \alpha_2/\alpha_1$ . The general form of the invariant manifold is defined by the expression

$$W\left(\frac{r}{t^\delta}, \frac{v}{t^{\delta-1}}, \frac{\rho}{t^{2(\delta-1)/(\alpha-1)}}, T\right) = 0,$$

where  $W$  is an arbitrary function.

The change of variables in Eq. (42) is governed by the formulae

$$v = t^{\delta-1} \tilde{v}(\xi), \quad \rho = t^{2(\delta-1)/(\kappa-1)} \rho_0 R(\xi), \quad T = \theta(\xi), \quad \xi = \frac{r}{t^\delta}, \tag{45}$$

In this case, similarly to the previous,  $\theta(\xi) = c_1 = \text{const}$ . The system of differential equations in the invariant variables is

$$\begin{aligned} (v - \delta\xi) \frac{dv}{d\xi} + NLR^{\kappa-2} \frac{dR}{d\xi} + (\delta - 1)v &= 0, \\ R \frac{dv}{d\xi} + (v - \delta\xi) \frac{dR}{d\xi} + 2 \frac{\delta - 1}{\kappa - 1} R + v \frac{Rv}{\xi} &= 0, \end{aligned} \tag{46}$$

where  $N = \phi(c_1)$ .

**Case IV.** For self-similar solutions exist it is necessary that  $\phi(z) = Az^\mu$  and  $\frac{d}{dz}\theta(z) = \kappa Bz^{\kappa-1}$ . The parameter  $\lambda$  in (36) is arbitrary, and  $\tau$  meets the condition

$$\tau = \frac{2(\beta - \alpha) - \mu\lambda}{\kappa - 1}. \tag{47}$$

The invariant manifold of group (36) is defined in this case by the equation

$$W\left(\frac{r}{t^\delta}, \frac{v}{t^{\delta-1}}, \frac{\rho}{t^\eta}, \frac{T}{t^\sigma}\right) = 0,$$

where  $\eta = \frac{2(\delta-1) - \mu\sigma}{\kappa-1}$  and  $\sigma$  is an arbitrary parameter. The transition to an equation in automodel variables is provided by the change

$$v = t^{\delta-1} \tilde{v}(\xi), \quad \rho = t^\eta \rho_0 R(\xi), \quad T = t^\sigma \theta(\xi), \quad \xi = \frac{r}{t^\delta}, \tag{48}$$

substituting (48) into (35) we obtain the system

$$(\tilde{v} - \delta\xi) \frac{d\tilde{v}}{d\xi} + \kappa B \frac{AR}{\rho_0} R^{\kappa-2} \frac{dR}{d\xi} + (\delta - 1)\tilde{v} = 0, \quad R \frac{d\tilde{v}}{d\xi} + (\tilde{v} - \delta\xi) \frac{dR}{d\xi} + \eta R + v \frac{\tilde{v}R}{\xi} = 0. \tag{49}$$

In the course of solving a system of equations in automodel variables for a problem of strong explosion it necessary to know another self-similar solution that matches the required one at the surface of strong discontinuity (the shock wave front). The abutting line must be the level curve  $\xi = \Lambda = \text{const}$ . If the adjoining  $\xi = \Lambda$  is performed along the line of strong discontinuity  $\xi_\phi$ , then on  $\xi_\phi = \Lambda$  the following relation for the velocity of the shock wave holds:

$$D = \delta \Lambda t^{\delta-1}.$$

The most simple is the case when near the shock wave front the velocity, the density and the temperature have constant (nonzero) values  $v_0, \rho_0$  and  $T_0$ , respectively. Direct verification shows that a solution near the shock wave front is self-similar if and only if  $v = 0, \delta = 1$  and  $\sigma = 0$ ; or, when  $v_0 = 0$ , if  $v = 0$  is arbitrary,  $\delta = 1$  and  $\sigma = 0$ . For strong explosion we can set  $\rho_0 = 0$  near the shock wave front. In this case a solution is self-similar if for  $\mu, \sigma$  and  $\delta$  the relation  $\mu\sigma = 2(\delta - 1)$  holds. The self-similarity indices for the solution on both sides of the surface of discontinuity must be the same.

The boundary conditions at the shock wave front arise from the momentum and mass conservation laws. Taking into account that near the shock wave front the pressure and the velocity of the medium vanish and  $p$  can be expressed via  $\rho$  and  $T$  by formula (2), we rewrite these conditions as

$$\begin{aligned} \rho_1 D^2 &= \rho_2 (v_2 - D)^2 + \rho_2, \\ \rho_2 (v_2 - D) + \rho_1 D &= 0, \end{aligned} \tag{50}$$

where the variables with the subscript  $_2$  characterize the medium state after the shock wave front, and those with the subscript  $_1$  are valid before it. Rewriting (50) in automodel variables we obtain the following boundary conditions under  $\Lambda = \xi_\phi = \text{const}$ :

$$\begin{aligned} R_2 (v_2 - \delta\Lambda) + \delta\Lambda &= 0, \\ R_1 (\delta\Lambda)^2 &= R_2 (v_2 - \delta\Lambda)^2 + \frac{AB}{\kappa} (R_2^\kappa - 1). \end{aligned} \tag{51}$$

From the expression for the total energy of the medium the value of the parameter  $\delta$  can be obtained. Suppose that the internal energy of the liquid is

$$\mathcal{E} = \frac{p}{\rho_0} F\left(\frac{\rho}{\rho_0}\right) \tag{52}$$

with a certain function  $F(z)$ . Then the total energy  $E_0$  is

$$E_0 = \Sigma(v) \int_0^{r_\Lambda} \left[ \frac{1}{2} \rho v^2 + \frac{p}{\rho_0} F\left(\frac{\rho}{\rho_0}\right) \right] r^\nu dr, \tag{53}$$

where  $r_\Lambda$  is the coordinate of the shock wave front. Expressing  $v$  and  $\rho$  in (53) via automodel variables, due to formula (50) we have

$$E_0 = t^{\delta(v+3)-2} \rho_0 \Sigma(v) \int_0^{\xi_\phi} \left[ \frac{R \tilde{v}^2}{2} + \frac{AB}{\kappa \rho_0} (R^\kappa - 1) F(R) \right] \xi^{v+1} d\xi.$$

Since the energy does not depend on  $t$  and in the limit  $t \rightarrow \infty$  it must take the finite value  $E_0$  ( $0 < E_0 < \infty$ ) it is necessary to demand

$$(v + 3)\delta - 2 = 0. \tag{54}$$

Let us return now to system (49). For  $\eta = 0$  it can be rewritten as

$$(\tilde{v} - \delta\xi) \frac{d\tilde{v}}{d\xi} + NR^{\kappa-2} \frac{dR}{d\xi} + (\delta - 1)\tilde{v} = 0, \quad R \frac{d\tilde{v}}{d\xi} + (\tilde{v} - \delta\xi) \frac{dR}{d\xi} + v \frac{R\tilde{v}}{\xi} = 0, \tag{55}$$

where  $N = \kappa AB / \rho_0$ . When  $\delta = 1/2$  ( $v = 1$ ) this system has a constant of motion. Indeed, the first equation yields

$$\frac{\partial}{\partial \xi} \left\{ \frac{\tilde{v}^2}{2} - \delta \xi \tilde{v} + \frac{N}{\kappa - 1} R^{\kappa-1} \right\} = (1 - 2\delta)\tilde{v} \equiv 0|_{\delta=\frac{1}{2}}.$$

Hence

$$R = \left\{ C_9 - \frac{\tilde{v}}{2} (\tilde{v} - \xi) \frac{\kappa - 1}{N} \right\}^{\frac{1}{\kappa-1}}. \tag{56}$$

Substituting (56) into the second equation of (55) we have an equation containing only the variables  $\xi, \tilde{v}$  and  $d\tilde{v}/d\xi$ :

$$\frac{d\tilde{v}}{d\xi} = \frac{\frac{\xi}{2} \tilde{v}(\xi - 2\tilde{v}) - v\tilde{v}[2C_9 - \tilde{v}(\tilde{v} - \xi)](\kappa - 1)}{\xi \left\{ (\kappa - 1)[2C_9 - \tilde{v}(\tilde{v} - \xi)] - \frac{1}{2}(\xi - 2\tilde{v})^2 \right\}}. \tag{57}$$

If  $C_9 = 1$  then via the substitution  $\tilde{v} = \xi y$  Eq. (57) transforms to

$$\frac{d\xi}{\xi} = - \frac{2(1 + \kappa)y^2 - 2(1 + \kappa)y + 1}{2(\kappa - 1)y^3 + 2y^2 - (1 + 2\kappa)y + 1} dy. \tag{58}$$

It is obvious that the expression in the r.h.s. of (58) can be integrated for any value of  $\kappa$ .

Now consider the case when  $\delta$  is arbitrary. Algebraic manipulations transform system (49) to the form

$$\begin{aligned} p \frac{dR}{d\xi} &= v(\tilde{v} - \delta\xi)R - \xi(\delta - 1)\tilde{v}R, \\ P \frac{d\tilde{v}}{d\xi} &= \xi(\tilde{v} - \delta\xi)(\delta - 1)v - vNR^{\kappa-1}\tilde{v}, \end{aligned} \tag{59}$$

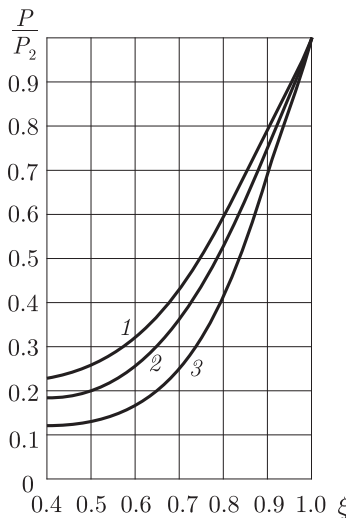


Fig. 1.

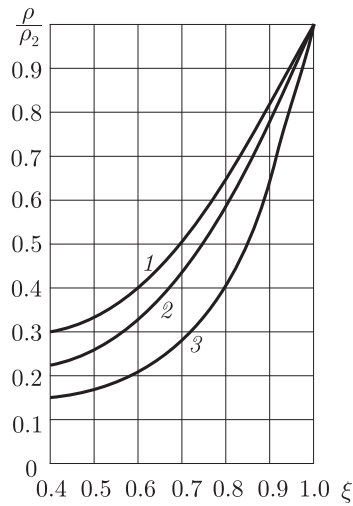


Fig. 2.

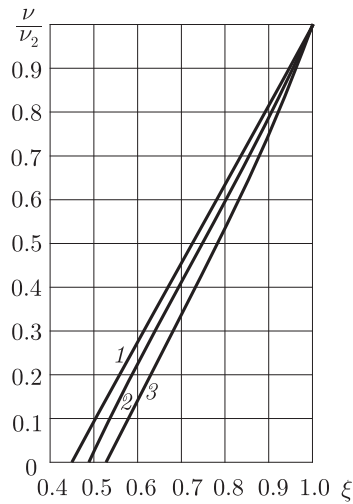


Fig. 3.

where  $P = \xi(NR^{\kappa-1} - (\tilde{v} - \delta\xi)^2)$ ,  $N = \kappa AB/\rho_0$  (supposing  $P \neq 0$ ). The substitution  $\tilde{v} = \xi x$ ,  $R^{\kappa-1} = \xi^2 y$  reduces the system to the form

$$\begin{aligned} 1 + \frac{\xi}{x} \frac{dx}{d\xi} &= \frac{\xi(x\xi - \delta\xi)(\delta - 1)x\xi - vN\xi^2 y \xi x}{xP}, \\ 2 + \frac{\xi}{y} \frac{dy}{d\xi} &= \frac{(\kappa - 1)\xi[v(x\xi - \delta\xi)\xi x - \xi(\delta - 1)\xi x]}{P}. \end{aligned} \tag{60}$$

From system (60) a single differential equation can be obtained, which relates  $x$  and  $y$ :

$$\frac{dx}{dy} = \frac{x}{y} \frac{(\delta - 1)(x - \delta) - vNy - [Ny - (x - \delta)^2]}{x(\kappa - 1)[v(x - \delta) - (\delta - 1)] - 2[Ny - (x - \delta)^2]}. \tag{61}$$

The boundary values  $x_2, y_2$  will be determined after substituting the expressions for  $\tilde{v}$  and  $R$  into (45). Simple manipulations yield

$$y_2 = \frac{1}{\Lambda^2} \left( \frac{\delta}{\delta - x_2} \right)^{\kappa-1}.$$

The value of  $x_2$  arises from the equation

$$\delta\Lambda^2(\delta - x_2)^{\kappa+1} - [M - (\delta\Lambda)^2](\delta - x_2)^\kappa + M\delta^\kappa = 0,$$

where  $M = AB/\rho_0$ .

Eq. (61) has irregularities when the denominator vanishes, namely for  $y = 0$  or  $x(x-1)[v(x-\delta) - (\delta-1)] - 2[Ny - (x-\delta)^2] = 0$ . In view of the definition of  $y$ , the first condition requires  $\xi = 0$  (it corresponds to the infinite  $r$  under  $t > 0$ ). However, we are interested in studying motion in a bounded domain, because when a perturbation spreads over long distances from the explosion site, then the assumption of self-similarity becomes unacceptable. Hence, we should consider the second condition. For the derivative  $dy/dx$  to be finite it is necessary that the numerator vanishes at the same time, i.e.  $x = 0$  or  $(\delta-1)(x-\delta) - vNy - 2[Ny - (x-\delta)^2] = 0$ .

Investigating the fields of integral curves of Eq. (61) we see that to solve the problem it is necessary to find the integral curve that contains irregular points and to determine the points of its intersection with the boundary curve.

This problem was solved using numerical integration. The results for all three types of symmetry are presented in Figs. 1–3.

The profiles of the physical variables in the domain covered by the shock wave are similar for all values of time. As it is shown in the graphs, the domain that surrounds the center of the explosion up to a distance approximately equal to one half of the shock wave radius, is the area of of the homogeneous state, where the velocity is zero.

The spread of the shock wave over time is determined as usual [1], by drawing up a the energy balance in the whole domain covered by the shock wave.

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