FUNCTIONAL
DIFFERENTIAL
EQUATIONS

VOLUME 20
2013, NO 3–4
PP. 225–235

APPLICATION OF THE AVERAGING METHOD TO SOME OPTIMAL CONTROL PROBLEMS

V. MOGILOVA *, O. STANZHYTSKYI † AND A. TKACHUK ‡

Abstract. In this paper we consider optimal control problem for non-linear differential equations system with small parameter. To the study of such problems the application of the averaging method is justified.

Key Words. Optimal Control, Small Parameter, Averaging Method, Infimum.

AMS(MOS) subject classification. 49N10, 49K27, 34B27, 34K10

1. Introdaction. One of the most widespread methods for an analysis of nonlinear dynamical systems is the averaging method. For ordinary differential equations this method was justified by Bogolyubov [1].

Later this method was generalized and developed for the various classes of differential equations, for example, such as pulse [2], functional-differential [3] and stochastic [4] equations.

Averaging method appeared to be also effective for the solving of optimal control problems. There is a number of papers, devoted to studying these issues (see, for example, [5]), where one could find an extensive bibliography. In the paper [6] a different from previously known approach is designed, regarding the use of the averaging method to the solution of optimal control problems. Namely, firstly an explicitly incoming time was averaged, while the control function was considered as a parameter, then studying the averaged system one was dealing with the same controls as for the original problem.

* National Technical University of Ukraine 'Kyiv Polytechnic Institute',Kiev, Ukraine, ostanzh@fmail.com

† Taras Shevchenko National University of Kyiv; National Technical University of Ukraine 'Kyiv Polytechnic Institute',Kiev, Ukraine, ostanzh@fmail.com

‡ National University of Food Technologies, Kyiv, Ukraine, tkachukam@ukr.net

Thus the averaged object was built in quite a simple manner, and at the same time the sets of admissible controls of the original and the averaged problems coincided.

In our paper, a generalization of the result [6] is obtained.

Firstly, Lipschitz condition for the right parts of control is replaced by the weaker condition of uniform continuity.

Secondly, and most importantly, there is provided a simpler averaged object - an autonomous system of differential equations.

The main result is an establishment of a connection between optimal controls of an averaged and an origin problems, and it is proved that the optimal control of the averaged system is ε-optimal for the origin system.

This paper consists of an introduction, which provides an overview of the previous results, statement of the problem, the lemma on the justification of the averaging method in the proper form for further studies and the main result. Also an example of applying this method to the study of optimal control of the oscillating system is presented.

2. Statement of the problem. We consider an optimal control problem of the ordinary differential equations system in the standard by Bogolyubov form.

$$\dot{x} = \varepsilon X(t, x, u),$$

$$x(0, u(0)) = x_0, \tag{1}$$

with quality criterion

$$\Phi(x(\frac{T}{\varepsilon}, u)) \to \inf \tag{2}$$

on the interval $[0, \frac{T}{\varepsilon}]$, where $\varepsilon > 0$ is a small parameter, $x \in D \subset \mathbb{R}^n$ is a phase vector, $u \in U \subset \mathbb{R}^m$ is a control vector, U is a closed set, $t \geq 0$, $T > 0$ is some constant, the function $X\colon \{t \geq 0\} \times D \times U \to \mathbb{R}^n$ is continuous on the set of variables.

In the following we observe $x(t, u)$ as a solution to the Cauchy problem (1) which correspond to the control $u = u(t)$.

The control $u(t)$ is admissible for the problem (1), (2), if next conditions hold:

a1 function $u(t)$ is measurable, locally integrable for $t \geq 0$;

a2 $u(t) \in U$ for $t \geq 0$;

a3 for each $u(t)$ there exists a constant u_0 such that $u(t) \to u_0$, $t \to \infty$, while such uniform convergence for each $u(t)$, that is: for every $\delta > 0$ there exists $T_0 > 0$, which is independent of $u(t)$ and u_0, that for every $t \geq T_0$ the inequality $|u(t) - u_0| < \delta$ holds;

a4 a solution of the Cauchy problem $x(t, u)$ defined on the interval $[0, \frac{T}{\varepsilon}]$ and for $[0, \frac{T}{\varepsilon}]$ belongs to the domain D for all $\varepsilon_0 \geq \varepsilon > 0$ for some $\varepsilon_0 > 0$.

We denote the set of admissible controls for the problem (1), (2) as F.

The problem of optimal control for the system (1) is to find such control $u \in F$ that provides minimal value of quality criterion (2). We observe $J_\varepsilon^* = \inf_{u \in F} J_\varepsilon(u)$.

We remark that the condition **a3** obviously holds if there exists an independent of $u \in F$ function $\varphi(t)$, which converges to zero for $t \to \infty$, such that $|u(t) - u_0| \leq \varphi(t)$.

The optimal control problem (1), (2) we put in correspondence to the next averaged optimal control problem

$$\begin{cases} \dot{y} = \varepsilon X_0(y, u_0) \\ y(0, u_0) = x_0 \end{cases} \tag{3}$$

with quality criterion

$$\Phi(y(\frac{T}{\varepsilon}, u_0)) \to \inf \tag{4}$$

, where vector $u_0 \in U$ is constant and

$$X_0(y, u_0) = \lim_{T \to \infty} \frac{1}{T} \int_0^T X(t, y, u_0) dt. \tag{5}$$

The control for the problem (3), (4) is admissible, if next conditions hold:

b1 $u_0 \in U$;

b2 a solution of the Cauchy problem (3) $y(t, u_0)$ defined on the interval $[0, \frac{T}{\varepsilon}]$ and for $[0, \frac{T}{\varepsilon}]$ belongs to the domain D for all $\varepsilon_0 > \varepsilon > 0$;

We denote the set of admissible controls for the problem (3), (4) as $\bar{F}$. Also we denote $\bar{J}_\varepsilon^* = \inf_{u_0 \in \bar{F}} \bar{J}_\varepsilon(u_0)$.

Let $u_0^*(\varepsilon)$ be an averaged optimal control for the problem (3), (4), that is $\bar{J}_\varepsilon(u_0^*(\varepsilon)) = \bar{J}_\varepsilon^*$. In our paper we prove that this control is η-optimal for the origin problem (1), (2), that is for any $\eta > 0$ there exists $\varepsilon_0 > 0$ such that for $0 < \varepsilon < \varepsilon_0$ inequality $|J_\varepsilon(u_0^*(\varepsilon)) - J_\varepsilon^*| < \eta$ holds.

3. Averaging Lemma. To obtain above mentioned result we need to prove lemma, which is a generalization of averaging principle in a case of right parts depend on functional parameters. This lemma also generalizes corresponding result from [6].

LEMMA 1. *Let the next conditions hold for domain* $Q = \{x \in D, t \geq 0, u \in U\}$*:*

1. *function* $X(t, x, u)$ *is measurable on the set of variables;*
2. *there exists constant* $K > 0$ *such that* $|X(t, x, u)| \leq K$ *for* $t \geq 0$, $x \in D$, $u \in U$*;*
3. *function* $X(t, x, u)$ *in domain* Q *satisfies Lipschitz condition on a variable* x *with constant* M*:* $|X(t, x, u) - X(t, x_1, u)| \leq M|x - x_1|$ *for any* $t \geq 0$, $x, x_1 \in D$, $u \in U$*;*
4. *function* $X(t, x, u)$ *is uniformly continuous regarding* $t \geq 0$, $x \in D$ *on* $u \in U$*;*
5. *solution* $y = y(t, u_0)$, $y(0, u_0) = x_0$ *of the averaged problem* (3) *defined for every* $t \geq 0$, $u_0 \in U$ *and belongs to the domain* D *along with some* ρ*-neighborhood, which is independent of* u_0*;*
6. *limit* (5) *exists uniformly regarding* $x \in D$ *and* $u_0 \in U$.

Then for any $\eta > 0$ *and* $T > 0$ *there exists* $\varepsilon_0 = \varepsilon_0(\eta, T) > 0$ *such that for any* $0 < \varepsilon < \varepsilon_0$ *solution* $x(t, u)$ *of the Cauchy problem for origin system defined on* $[0, \frac{T}{\varepsilon}]$ *and an estimation*

$$|x(t, u) - y(t, u_0)| \leq \eta \tag{6}$$

is valid for each control $u(t)$*, which satisfies conditions* **a1**$-$**a3***, while control* u_0 *is from condition* **a4**.

REMARK 1. *This lemma states that the estimation* (6) *is uniform over every control* $u(t)$*, which satisfies conditions* **a1**-**a3**.

Proof. We take any $\eta > 0$ such that $\eta < \frac{\rho}{2}$ and fixate it. For $\varepsilon > 0$ and any admissible control $u(t)$ we evaluate on $[0, \frac{T}{\varepsilon}]$ the norm of the difference between solutions of the system (1) and next system

$$\begin{cases} \dot{\bar{x}} = \varepsilon X_0(t, \bar{x}, u_0) \\ \bar{x}(0) = x_0 \end{cases} \tag{7}$$

$\bar{x}(t) = \bar{x}(t, u_0)$, where u_0 is selected from condition **a3** for $u(t)$.

Now passing in (1) and (7) to the integral representation, we have for $t \geq 0$ to the moment of exiting at least one of solution on a domain D boundary

$$x(t) = x_0 + \varepsilon \int_0^t X(s, x(s), u(s))ds, \tag{8}$$

$$\bar{x}(t) = x_0 + \varepsilon \int_0^t X(s, \bar{x}(s), u_0) ds, \tag{9}$$

where $x(t) = x(t, u)$.

From (8) and (9) we obtain

$$x(t) - \bar{x}(t) = \varepsilon \int_0^t (X(s, x(s), u_0) - X(s, \bar{x}(s), u_0)) ds +$$

$$+\varepsilon \int_0^t (X(s, x(s), u(s) - X(s, x(s), u_0))) ds, \tag{10}$$

Lemma's condition 4 implies that there exists $\delta > 0$ such that if $|u - u_0| < \delta$ then

$$|X(t, x, u) - X(t, x, u_0)| < \frac{\eta}{2T \exp\{TM\}} \tag{11}$$

for $t \geq 0$, $x \in D$, while δ does not depend on t and x.

We choose T_0 from condition **a4** in such a way that for $t \geq T_0$ the following inequality holds

$$|u(t) - u_0| < \delta. \tag{12}$$

It is obvious that we consider ε such that $\frac{T}{\varepsilon} \geq T_0$.

In a view of Lemma's conditions, from (10)- (12) we get

$$|x(t) - \bar{x}(t)| \leq \varepsilon M \int_0^t |x(s) - \bar{x}(s)| ds +$$

$$+\varepsilon \int_0^{T_0} |X(s, x(s), u(s)) - X(s, x(s), u_0)| ds +$$

$$+\varepsilon \int_{T_0}^t |X(s, x(s), u(s) - X(s, x(s), u_0)| ds \leq$$

$$\leq \varepsilon M \int_0^t |x(s) - \bar{x}(s)| ds + \varepsilon 2KT_0 + \frac{\eta}{2 \exp\{TM\}}.$$

Due to Gronwall-Bellman Lemma we obtain

$$|x(t) - \bar{x}(t)| \leq (\varepsilon 2KT_0 + \frac{\eta}{2 \exp\{TM\}}) \exp\{TM\}. \tag{13}$$

For solutions of systems (3) and (7), due to Theorem 1 ([7], p.10), for $0 < \varepsilon < \varepsilon_1$ next estimation is valid for $t \in [0, \frac{T}{\varepsilon}]$

$$|\bar{x}(t) - y(t, u_0)| \leq \frac{\eta}{2}. \tag{14}$$

In this case $\bar{x}(t)$ defined for $t \in [0, \frac{T}{\varepsilon}]$ and belongs with its $\frac{5\rho}{6}$-neighborhood to the domain D.

From (13) and (14) we obtain

$$|x(t, u) - y(t, u_0)| \leq |x(t, u) - \bar{x}(t)| + |\bar{x}(t) - y(t, u_0)|$$

$$\leq \varepsilon 2KT_0 \exp\{TM\} + \frac{\eta}{2}. \tag{15}$$

Now if we take $\varepsilon_2 = \frac{\eta}{2KT_0 \exp\{TM\}}$, then for all $0 < \varepsilon < \varepsilon_0$, where $\varepsilon_0 = \min \varepsilon_1, \varepsilon_2$, estimation (6) holds.

The Lemma is proved. □

4. The Main Result. Let us proceed to the main result. The following theorem takes place.

THEOREM 1. *Let in the domain $Q = \{x \in D, t \geq 0, u \in U\}$ next conditions hold:*

1. *function $X(t, x, u)$ is measurable on the set of variables, bounded by constant K and satisfies Lipschitz condition over variable x with constant M;*
2. *function $X(t, x, u)$ is uniformly continuous regarding $t \geq 0$, $x \in D$ on $u \in U$;*
3. *function $\Phi(x)$ in domain D satisfies Lipschitz condition on a variable x with constant L;*
4. *limit (5) exists uniformly regarding $x \in D$ and $u_0 \in U$;*
5. *solution $y = y(t, u_0)$, $y(0, u_0) = x_0$ of the averaged problem (3) defined for every $t \geq 0$, $u_0 \in U$ and belongs to the domain D along with some ρ-neighborhood, which is independent of u_0;*
6. *there exists optimal control $u_0^*(\varepsilon)$ of the problem (3), (4) in from the class $\bar{F}$.*

Then for any $\eta > 0$ there exists $\varepsilon_0 = \varepsilon_0(\eta)$ such that

a *for any $0 < \varepsilon < \varepsilon_0$ inequality $J_\varepsilon^* > -\infty$ is valid;*

b *the subsequent inequality holds*

$$|J_\varepsilon(u_0^*(\varepsilon)) - J_\varepsilon^*| \leq \eta. \tag{16}$$

Namely optimal control of the averaged problem is η-optimal for the origin problem.

Proof. Firstly we show that for the problem (1), (2) we have $J_\varepsilon^* = \inf_{u\in F} J_\varepsilon(u) > -\infty$.

Let us assume this statement is false. Then there exists consequence $\{\varepsilon_n\}$ such that $\varepsilon_n \to 0$, $n \to \infty$ and

$$J_{\varepsilon_n}^* = -\infty. \tag{17}$$

For any ε_n from the definition of the infimum it implies an existence of admissible controls sequence $\{u_m^n\}$ such that for $m \to \infty$

$$J_{\varepsilon_n}(u_m^n) \to -\infty. \tag{18}$$

We observe $x_m^n(t) = x(t, u_m^n)$ as solution of the system (1) for control u_m^n, and $y_m^n(t) = y(t, u_{0,m}^n)$ as solution of the averaged system (3) for control $u_{0,m}^n$, where $u_{0,m}^n = \lim_{t\to\infty} u_m^n$ is from the condition **a3**. Due to U is closed, we have $u_{0,m}^n \in U$. Also we have $J_\varepsilon(u_m^n) = \Phi(x_m^n(\frac{T}{\varepsilon_n}))$.

Since foe every ε there exists an optimal control of the problem (3), (4), then

$$\bar{J}_\varepsilon(u_{0,m}^n) = \Phi(\bar{y}_m^n(\frac{T}{\varepsilon_n})) \geq \bar{J}_{\varepsilon_n}^* > -\infty.$$

Now we fixate $0 < \eta_0 < \frac{\varrho}{2}$. From the above it follows an existence of integer n_0 such that for $0 < \varepsilon_n < \varepsilon_{n_0}$ next estimations are valid

$$|J_{\varepsilon_n}(u_m^n) - \bar{J}_{\varepsilon_n}(u_{0,m}^n)| = |\Phi(x_m^n(\frac{T}{\varepsilon_n})) - \Phi(y(\frac{T}{\varepsilon_n}))| \leq$$

$$\leq L|x_m^n(\frac{T}{\varepsilon_n}) - y(\frac{T}{\varepsilon_n})| \leq L\eta_0.$$

Further we have

$$J_{\varepsilon_n}(u_m^n) = J_{\varepsilon_n}(u_m^n) - \bar{J}_{\varepsilon_n}(u_{0,m}^n) + \bar{J}_{\varepsilon_n}(u_{0,m}^n) \geq$$

$$\geq \bar{J}_{\varepsilon_n}^* - L\eta_0 > -\infty$$

for all m, which leads to a contradiction with (18), and with (17) as a result. Hence statement **a** of the theorem is proved.

Now we prove the statement **b**. For this purpose we write down such inequality $J_\varepsilon^* \leq \bar{J}^\varepsilon + (J_\varepsilon(u_0^*) - \bar{J}^\varepsilon)$.

We note that control u_0^* is optimal for the origin problem (1), (2).

Let us evaluate subsequent difference

$$|J_\varepsilon(u_0^*) - \bar{J}_\varepsilon(u_0^*)| = |\Phi(x(\frac{T}{\varepsilon}, u_0^*)) - \Phi(y(\frac{T}{\varepsilon}, u_0^*))| \leq$$

$$\leq L|x(\frac{T}{\varepsilon}, u_0^*) - y(\frac{T}{\varepsilon}, u_0^*)|.$$

Applying above proved Lemma for any $0 < \eta_1 < \frac{\varrho}{2}$ and all small enough ε, we obtain an estimation

$$J_\varepsilon^* \leq \bar{J}_\varepsilon + L\eta_1. \tag{19}$$

Further from the definition of the infimum we have that for chosen $\eta_1 > 0$ there exists control $u_{\eta_1}(t,\varepsilon)$ such that

$$J(u_{\eta_1}(t,\varepsilon)) < J_\varepsilon^* + \eta_1. \tag{20}$$

Let $u_{0,\eta} = \lim_{t\to\infty} u_{\eta_1}(t,\varepsilon)$ be from the condition **a3**. Then next estimation is valid

$$\bar{J}_\varepsilon^* = \bar{J}_\varepsilon(u_0^*) \leq \bar{J}_\varepsilon(u_{0,\eta_1}) \leq$$
$$\leq \bar{J}_\varepsilon(u_{0,\eta_1}) + J_\varepsilon^* + \eta_1 - J(u_{\eta_1}(t,\varepsilon)).$$

Once again we use Lemma so as to obtain

$$|\bar{J}_\varepsilon(u_{0,\eta_1}) - J(u_{\eta_1}(t,\varepsilon))| = |\Phi(y(\frac{T}{\varepsilon}, u_{0,\eta_1})) - \Phi(x(\frac{T}{\varepsilon}, u_{0,\eta_1}(t,\varepsilon))| \leq$$

$$\leq L|y(\frac{T}{\varepsilon}, u_{0,\eta_1}) - x(\frac{T}{\varepsilon}, u_{0,\eta_1}(t,\varepsilon))| \leq L\eta_1.$$

Thus, we have $\bar{J}_\varepsilon^* \leq J_\varepsilon^* + (L+1)\eta_1$, which in a view of (19) implies that

$$|J_\varepsilon^* - \bar{J}_\varepsilon^*| \leq (L+1)\eta_1. \tag{21}$$

Further we consider the difference

$$|J_\varepsilon(u_0^*) - \bar{J}_\varepsilon^*| = |\Phi(x(\frac{T}{\varepsilon}, u_0^*)) - \Phi(y(\frac{T}{\varepsilon}, u_0^*))| \leq$$

$$\leq L|x(\frac{T}{\varepsilon}, u_0^*) - y(\frac{T}{\varepsilon}, u_0^*)| \leq L\eta_1. \tag{22}$$

The last estimation follows from the same arguments as the estimation (14).

Onwards due to inequalities (21) and (22), finally we have

$$|J_\varepsilon(u_0^*) - J_\varepsilon^*| \le \eta,$$

where $\eta \equiv \eta_1(2L+1)$.
This proves the Theorem. ∎

To the theorem proved, we make two remarks.
REMARK 2. *If in the theorem's conditions we state that admissible set U is compact and function $X_0(y, u_0)$ is continuous on the set of variables, then condition* **6** *will be fulfilled.*

Indeed, in this case, by the theorem on the continuous dependence of solutions of differential equations in the parameter, we have that function $\bar{J}_\varepsilon(u_0)$ is continuous on u_0 and defined on a compact. Consequently, by the Weierstrass theorem, this function function the minimum value.
REMARK 3. *If the function $X_0(y, u_0)$ is continuously differentiable on the variables $y \in D$, $u_0 \in U$, then the solution $y(t, u_0)$ of the equation* (3) *is smooth on u_0. If in addition function $\Phi(z)$ is smooth, then the optimization problem* (4) *is a finite smooth problem. Therefore minimal value of this problem lies on the boundary of the set U or in the inner points, where*

$$\frac{d\Phi(y(\frac{T}{\varepsilon}, u_0))}{du_0} = 0 \tag{23}$$

The last equation we could rewrite in the form of

$$\frac{\partial \Phi}{\partial y}\frac{\partial y(\frac{T}{\varepsilon}, u_0)}{\partial u_0} = 0 \tag{24}$$

In addition $\frac{\partial y(t,u_0)}{\partial u_0}$ satisfies the variational equation

$$\frac{d}{dt}\frac{\partial y(t, u_0)}{\partial u_0} = \varepsilon \frac{\partial X_0(y(t, u_0), u_0)}{\partial y}\frac{\partial y(t, u_0)}{\partial u_0} + \frac{\partial X_0(y(t, u_0), u_0)}{\partial u_0}$$

and initial condition $\frac{\partial y(0,u_0)}{\partial u_0} = 0$.

This two remarks often allow to reduce the optimal control problem to finite smooth problem. The last allows to solve problem (3), (4) in simpler methods, than dynamic programming method or the maximum principle, and consequently it allows relatively easy to find "almost" optimal control for the origin problem (1), (2).

Example We consider optimal control problem with an oscillating object, which describes by nonlinear equation of the second order

$$\ddot{x} + x = \varepsilon(-2\dot{x} + x^3 + 2\dot{x}u),$$

$$x(0) = x_0, \quad \dot{x}(0) = x_1, \quad |x_0| + |x_1| < 1, \quad |u| \le 1, \quad L = \frac{T}{\varepsilon},$$

(25)
$$J_\varepsilon(u) = \frac{1}{2}(x^2(L) + \dot{x}^2(L)) \to \inf .$$

Amplitude– phase replacement

$$x = a\sin(t+\varphi),$$

$$\dot{x} = a\cos(t+\varphi)$$

leads (25) to the form of (1)

(26)
$$\begin{cases} \dot{a} = \varepsilon[-2a\cos(t+\varphi) + a^3\sin^3(t+\varphi) + 2ua\cos(t+\varphi)]\cos(t+\varphi), \\ \dot{\varphi} = \varepsilon[-2\cos(t+\varphi) + a^2\sin^3(t+\varphi) + 2u\cos(t+\varphi)]\sin(t+\varphi). \end{cases}$$

Herewith the quality functional is of the form

$$J_\varepsilon(u) = \frac{1}{2}a^2(L) \to \inf .$$

The averaged problem for (26) is

(27)
$$\begin{cases} \frac{d\xi}{d\tau} = \xi(u_0 - 1), \\ \frac{d\eta}{d\tau} = -\frac{3}{8}\xi^2. \end{cases}$$

$$\xi(0) = a_0, \quad \eta(0) = \varphi_0,$$

$$\bar{J}_\varepsilon(u_0) = \frac{1}{2}\xi^2(L) \to \inf, \quad \tau = \varepsilon t.$$

Since functional depends alone on variable ξ, then it is enough to consider only first equation of the system (27).

The solution of the Cauchy problem is of the form $\xi = a_0\exp\{(u_0-1)t\}$, and the quality functional is the following $\xi^2(L) = a_0^2\exp\{2(u_0-1)L\}$.

It follows from this that $u_0^* = -1$ is optimal control, which is also η-optimal for the (25).

Acknowledgements

This work is partially supported by the FFR of Ukraine under Grant No. 0113U003068.

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