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*Lyubov V. Kulavets', Oksana M. Mulyava and Myroslav M. Sheremeta*ON BELONGING OF CHARACTERISTIC FUNCTIONS  
OF PROBABILITY LAWS TO A CONVERGENCE CLASS**Summary**

It is investigated the condition on a probability law, under which its characteristic function belongs to a convergence class.

*Keywords and phrases:* probability law, characteristic function, convergence class

**1. Introduction**

A non-decreasing function  $F$  continuous on the left on  $(-\infty, +\infty)$  is said [1, p. 10] to be a *probability law* if

$$\lim_{x \rightarrow +\infty} F(x) = 1 \quad \text{and} \quad \lim_{x \rightarrow -\infty} F(x) = 0,$$

and the function

$$\varphi(z) = \int_{-\infty}^{+\infty} e^{izx} dF(x)$$

defined for real  $z$  is called [1, p. 12] a *characteristic function* of this law. If  $\varphi$  has an analytic continuation on the disk  $\mathbb{D}_R = \{z : |z| < R\}$ ,  $0 < R \leq +\infty$ , then we call  $\varphi$  a characteristic function of the law  $F$  analytic in  $\mathbb{D}_R$ . Further we always assume that  $\mathbb{D}_R$  is the maximal disk of the analyticity of  $\varphi$ . It is known [1, p. 37-38] that  $\varphi$  is a characteristic function of the law  $F$  analytic in  $\mathbb{D}_R$  if and only if for every  $r \in [0, R)$

$$(1) \quad W_F(x) =: 1 - F(x) + F(-x) = O(e^{-rx}), \quad x \rightarrow +\infty.$$

Hence it follows that

$$(2) \quad \lim_{x \rightarrow +\infty} \frac{1}{x} \ln \frac{1}{W_F(x)} = R.$$

For  $0 \leq r < R$  we put  $M(r, \varphi) = \max\{|\varphi(z)| : |z| = r\}$ . If a characteristic function  $\varphi \neq \text{const}$  is entire then [1, p. 45] there exists

$$\lim_{r \rightarrow +\infty} r^{-1} \ln M(r, \varphi) > 0,$$

that is  $\varphi$  has the growth not below of normal type of the order  $\varrho = 1$ . Therefore, if we define a convergence class [2, p. 62] by condition

$$(3) \quad \int_{r_0}^{+\infty} r^{-(\varrho+1)} \ln M(r, \varphi) dr < +\infty,$$

we need to assume that  $\varrho > 1$ . For the function  $\varphi$  analytic in  $\mathbb{D} = \mathbb{D}_1$  of order

$$\varrho = \overline{\lim}_{r \uparrow 1} \frac{\ln \ln M(r, \varphi)}{-\ln(1-r)} > 0$$

a *convergence class* is defined [3] by the condition

$$(4) \quad \int_{r_0}^1 (1-r)^{\varrho-1} \ln M(r, \varphi) dr < +\infty.$$

As  $\varrho = 2$  this condition is sufficient [4, p. 50] in order that  $\varphi$  belongs to the class of MacLane.

In the paper [5] it is asserted that an entire characteristic function  $\varphi$  of the order  $\varrho > 1$  of the probability law  $F$  belongs to convergence class if and only if

$$(5) \quad \int_{x_0}^{+\infty} \left( \frac{1}{x} \ln \frac{1}{W_F(x)} \right)^{1-\varrho} dx < +\infty.$$

For a characteristic function  $\varphi$  of the order  $\varrho > 0$  analytic in  $\mathbb{D}$  in [6] is proved that in order that  $\varphi$  belongs to convergence class it is necessary and, in the case when the function

$$v(x) = \ln \frac{1}{W_F(x)}$$

is continuously differentiable and  $v'$  increases, it is sufficient that

$$(6) \quad \int_{x_0}^{+\infty} \left\{ \left( 1 + \frac{1}{x} \ln W_F(x) \right)^+ \right\}^{\varrho+1} dx < +\infty.$$

Here we examine a problem of the belonging of analytic characteristic function of probability law to a convergence  $\Phi$ -class, which is considered in the papers [7–9]. For this purpose we have to use the next lemma.

**Lemma 1.** *If  $\varphi$  is a characteristic function of probability law  $F$  analytic in  $\mathbb{D}_R$ ,  $0 < R \leq +\infty$  then for every  $r \in [0, R)$  and all  $x \geq 0$  the following inequalities are true:*

$$W_F(x)e^{xr} \leq 2M(r, \varphi)$$

and

$$M(r, \varphi) \leq r \int_0^{\infty} W_F(x) e^{xr} dx + 1 + W_F(+0).$$

For  $R = +\infty$  these inequalities are proved in [1, pp. 54–55] and for  $R < +\infty$  the proof is analogous.

We put

$$\mu(r, \varphi) = \sup\{W_F(x)e^{xr} : x \geq 0\}$$

and

$$I(r, \varphi) = \int_0^{\infty} W_F(x) e^{xr} dx$$

and suppose that  $M(r, \varphi) \uparrow +\infty$  as  $r \uparrow R$ . Then by Lemma 1

$$(7) \quad \ln \mu(r, \varphi) \leq (1 + o(1)) \ln M(r, \varphi) \leq (1 + o(1)) \ln I(r, \varphi), \quad r \uparrow R.$$

Therefore, the problem of belonging of  $\ln M(r, \varphi)$  to that or other convergence class is transferred to the problem of belonging of  $\ln \mu(r, \varphi)$  to this convergence class and the estimates  $I(r, \varphi)$  by  $\mu(r, \varphi)$ .

## 2. Properties of the function $\ln \mu(r, \varphi)$

It is clear that  $W_F(x) \searrow 0$  ( $x \rightarrow +\infty$ ). Therefore, if  $R = +\infty$  and  $x_0 = \sup\{x : W_F(x) > 0\} < +\infty$  then

$$\mu(r, \varphi) = \sup\{W_F(x)e^{xr} : 0 \leq x \leq x_0\} \leq W_F(0)e^{x_0 r}$$

and since  $\mu(r, \varphi) \geq W_F(x_0)e^{x_0 r}$  we have

$$\lim_{r \rightarrow +\infty} r^{-1} \ln \mu(r, \varphi) = x_0.$$

If  $R = +\infty$  and  $W_F(x) > 0$  for all  $x \geq 0$  then

$$\lim_{r \rightarrow +\infty} r^{-1} \ln \mu(r, \varphi) = +\infty.$$

If  $R < +\infty$  then the function  $\ln \mu(r, \varphi)$  may be bounded. It is easy to show that  $\mu(r, \varphi) \leq K < +\infty$  for all  $r \in [0, R)$  if and only if  $W_F(x)e^{xR} \leq K < +\infty$  for all  $x \geq 0$ . Thus,  $\mu(r, \varphi) \uparrow +\infty$  as  $r \uparrow R$  if and only if

$$(8) \quad \overline{\lim}_{x \rightarrow +\infty} W_F(x)e^{xR} = +\infty.$$

From the definition of  $\mu(r, \varphi)$  it follows that the function  $\ln \mu(r, \varphi)$  is convex on  $[0, R)$ .

For  $r \in [0, R)$  and  $\varepsilon > 0$  we put

$$\nu(r, \varphi; \varepsilon) = \sup\{x \geq 0 : \ln W_F(x) + rx \geq \ln \mu(r, \varphi) - \varepsilon\}.$$

Clearly, for fixed  $r \in [0, R)$  the function  $\nu(r, \varphi; \cdot)$  is nondecreasing on  $(0, +\infty)$ . Therefore, there exists a quantity

$$\nu(r, \varphi) = \lim_{\varepsilon \downarrow 0} \nu(r, \varphi; \varepsilon).$$

**Lemma 2.** *The function  $\nu$  is nondecreasing on  $[0, R)$  and  $(\ln \mu(r, \varphi))' = \nu(r, \varphi)$  for all  $r \in (0, R)$  with the exception of an at most countable set.*

*Proof.* At first we prove that for arbitrary  $r \in (0, R)$  and  $\varepsilon > 0$  the set

$$E(r, \varepsilon) = \{x \geq 0 : |x - \nu(r, \varphi)| < \varepsilon, \ln W_F(x) + rx \geq \ln \mu(r, \varphi) - \varepsilon\}$$

is non-empty. Indeed, for fixed  $r \in (0, R)$  and  $\varepsilon > 0$  there exists  $\delta \in (0, \varepsilon)$  such that

$$(9) \quad |\nu(r, \varphi) - \sup\{x \geq 0 : \ln W_F(x) + \sigma x \geq \ln \mu(r, \varphi) - \delta\}| < \varepsilon/2$$

and there exists  $x_0 \geq 0$  such that

$$(10) \quad \ln f(x_0) + \sigma x_0 \geq \ln \mu(r, \varphi) - \delta$$

and

$$(11) \quad |x_0 - \sup\{x \geq 0 : \ln f(x) + rx \geq \ln \mu(r, \varphi) - \delta\}| < \varepsilon/2.$$

From (9)–(11) we obtain

$$|x_0 - \nu(r, \varphi)| < \varepsilon, \quad \ln W_F(x_0) + rx_0 \geq \ln \mu(r, \varphi) - \varepsilon,$$

that is  $x_0 \in E(r, \varepsilon)$  and the set  $E(r, \varepsilon)$  is non-empty.

Since  $E(r, \varepsilon)$  is non-empty for arbitrary  $r \in (0, R)$  and  $\varepsilon > 0$ , by the axiom of choice for each  $\varepsilon > 0$  there exists a function  $\nu_\varepsilon(r) \geq 0$  such that for all  $r \in (0, R)$

$$(12) \quad \ln \mu(r, \varphi) \geq \ln W_F(\nu_\varepsilon(r)) + r\nu_\varepsilon(r) \geq \ln \mu(r, \varphi) - \varepsilon$$

and

$$(13) \quad |\nu_\varepsilon(r) - \nu(r, \varphi)| < \varepsilon.$$

Suppose that  $r_1, r_2 \in (0, R)$ . By definition

$$(14) \quad \ln \mu(r_1, \varphi) \geq \ln W_F(\nu_\varepsilon(r_2)) + r_1\nu_\varepsilon(r_2).$$

From (12) we have

$$(15) \quad \ln \mu(r_2, \varphi) \leq \varepsilon + \ln W_F(\nu_\varepsilon(r_2)) + r_2\nu_\varepsilon(r_2),$$

and combining (14) and (15) we obtain

$$\ln \mu(r_2, \varphi) - \ln \mu(r_1, \varphi) \leq (r_2 - r_1)\nu_\varepsilon(r_2) + \varepsilon,$$

whence passing to the limit as  $\varepsilon \rightarrow 0$  and taking into account (13) we get

$$(16) \quad \ln \mu(r_2, \varphi) - \ln \mu(r_1, \varphi) \leq (r_2 - r_1)\nu(r_2, \varphi).$$

Since  $r_1$  and  $r_2$  are arbitrary, we can interchange them and obtain also the inequality

$$(17) \quad \ln \mu(r_1, \varphi) - \ln \mu(r_2, \varphi) \leq (r_1 - r_2)\nu(r_1, \varphi).$$

Suppose that  $r_1 < r_2$ . Then from (16) and (17) we get

$$(18) \quad \nu(r_1, \varphi) \leq \frac{\ln \mu(r_2, \varphi) - \ln \mu(r_1, \varphi)}{r_2 - r_1} \leq \nu(r_2, \varphi).$$

Inequality (18) implies that the function  $\nu(r, \varphi)$  is nondecreasing and, thus, continuous with the exception of an at most countable set of points. Passing to the limit in (18) as  $r_1 \rightarrow r_2$  (and afterwards  $r_2 \rightarrow r_1$ ), we obtain the equality  $(\ln \mu(r, \varphi))' = \nu(r, \varphi)$ . Lemma 2 is proved.

This lemma implies the following result.

**Corollary 1.** For all  $0 < r_0 < r < R$

$$(19) \quad \ln \mu(r, \varphi) = \ln \mu(r_0, \varphi) + \int_{r_0}^r \nu(x, \varphi) dx.$$

From (19) it follows that if  $\mu(r, \varphi) \uparrow +\infty$  as  $r \uparrow R$ ; then  $\nu(r, \varphi) \nearrow +\infty$  as  $r \uparrow R$ , and (12) implies for each  $\varepsilon > 0$  and all  $r \in (0, R)$  the inequality

$$\frac{1}{\nu_\varepsilon(r, \varphi)} \ln \frac{1}{W_F(\nu_\varepsilon(r, \varphi))} \leq r - \frac{\ln \mu(r, \varphi)}{\nu_\varepsilon(r, \varphi)} \leq r$$

whence in view of the arbitrariness of  $\varepsilon$  we obtain

$$(20) \quad \frac{1}{\nu(r, \varphi)} \ln \frac{1}{W_F(\nu(r, \varphi))} \leq r.$$

By  $V(R)$  we denote a class of positive continuously differentiable on  $(0, +\infty)$  function  $v$  such that  $v'(x) \uparrow R$  as  $x \rightarrow +\infty$ . If

$$\ln \frac{1}{W_F(x)} = v(x) \in V(R)$$

then for every  $r \in (0, R)$  the function  $\ln W_F(x) + rx = -v(x) + rx$  has a unique point of the maximum  $x = \nu(r, \varphi)$ , which is increasing and continuous on  $(0, R)$ , and

$$(21) \quad \ln \mu(r, \varphi) = \max\{\ln W_F(x) + rx : x \geq 0\} = \ln W_F(\nu(r, \varphi)) + r\nu(r, \varphi).$$

### 3. Belonging of $\ln \mu(r, \varphi)$ to a convergence $\Phi$ -class

Let  $0 < R \leq +\infty$  and  $\Omega(R)$  be a class of positive functions  $\Phi$  unbounded on  $(0, R)$  such that the derivative  $\Phi'$  is positive, continuously differentiable and increasing to  $+\infty$  on  $(0, R)$ . For  $\Phi \in \Omega(R)$  we denote by  $\phi$  the inverse function to  $\Phi'$ , and let

$$\Psi(r) = r - \frac{\Phi(r)}{\Phi'(r)}$$

be the function associated with  $\Phi$  in the sense of Newton. As in [7], it is possible to show that the function  $\Psi$  is continuously differentiable and increasing to  $R$  on  $(0, R)$  and the function  $\phi$  is continuously differentiable and increasing to  $R$  on  $(0, +\infty)$ .

As in [8–10], we say that  $\ln \mu(r, \varphi)$  belongs to a convergence  $\Phi$ -class if

$$(22) \quad \int_{r_0}^R \frac{\Phi'(r) \ln \mu(r, \varphi)}{\Phi^2(r)} dr < +\infty.$$

**Proposition 1.** *Let*

$$0 < R \leq +\infty, \quad \Phi \in \Omega(R)$$

and

$$(23) \quad \frac{\Phi''(r)\Phi(r)}{(\Phi'(r))^2} \geq h > 0, \quad r \in [r_0, R),$$

and  $\varphi$  be a characteristic function on probability law  $F$  analytic in  $\mathbb{D}_R$  such that

$$\ln \frac{1}{W_F(x)} = v(x) \in V(R).$$

If

$$(24) \quad \int_{x_0}^{\infty} \frac{dx}{\Phi' \left( \frac{1}{x} \ln \frac{1}{W_F(x)} \right)} < +\infty;$$

then  $\ln \mu(r, \varphi)$  belongs to a convergence  $\Phi$ -class.

*Proof.* At first we remark that the condition  $v(x) \in V(R)$  implies the correlation  $v(x) = (1 + o(1))xR$  as  $x \rightarrow +\infty$ , that is

$$\frac{1}{x} \ln \frac{1}{W_F(x)} = (1 + o(1))R \quad \text{as } x \rightarrow +\infty$$

and the condition (2) holds.

From (23) it follows that

$$\int_{r_0}^R \frac{dr}{\Phi(r)} \leq \frac{1}{h} \int_{r_0}^R \frac{\Phi''(r)dr}{(\Phi'(r))^2} = \frac{1}{h\Phi'(r_0)} < +\infty.$$

We put

$$B(x) = \int_x^R \frac{dr}{\Phi(r)}.$$

Then

$$B(x) \downarrow 0 \quad \text{as } x \uparrow R, \quad B(x) \leq \frac{1}{h\Phi'(x)},$$

and (24) implies

$$(25) \quad \int_{x_0}^{+\infty} B \left( \frac{1}{x} \ln \frac{1}{W_F(x)} \right) dx < +\infty.$$

From the condition

$$\ln \frac{1}{W_F(x)} = v(x) \in V(R)$$

it follows that there exists a function  $\nu(r) = \nu(r, \varphi)$  increasing and continuous on  $(0, R)$  such that (20) holds, and in view of (25) and decreasing of  $B$

$$(26) \quad \int_{r_0}^R B(r) d\nu(r) \leq \int_{r_0}^R B \left( \frac{1}{\nu(r)} \ln \frac{1}{W_F(\nu(r))} \right) d\nu(r) < +\infty.$$

But

$$+\infty > \int_{r_0}^R B(r) d\nu(r) = B(r)\nu(r) \Big|_{r_0}^R - \int_{r_0}^R \nu(r) B'(r) dr \geq \text{const} + \int_{r_0}^R \frac{\nu(r)}{\Phi(r)} dr.$$

Therefore, in view of (19)

$$\int_{r_0}^R \frac{\Phi'(r) \ln \mu(r, \varphi)}{\Phi^2(r)} dr = \int_{r_0}^R \ln \mu(r, \varphi) d \left( -\frac{1}{\Phi(r)} \right) \leq \text{const} + \int_{r_0}^R \frac{\nu(r, \varphi)}{\Phi(r)} dr < +\infty,$$

i.e.  $\ln \mu(r, \varphi)$  belongs to a convergence  $\Phi$ -class. Proposition 1 is proved.

*Remark 1.* We did not succeed to build an example indicative on importance of the condition

$$\ln \frac{1}{W_F(x)} = V(x) \in \Omega(+\infty).$$

It seems to us that it is superfluous, but for this purpose at least for the case of entire functions it is necessary to prove the following statement: *for every function  $v_1$  such that  $v_1(x) \nearrow +\infty$  and*

$$\frac{v_1(x)}{x} \rightarrow +\infty \quad \text{as } x \rightarrow +\infty$$

*there exists a function  $v \in V(R)$  such that  $v_1(x) = O(v(x))$  as  $x \rightarrow +\infty$  and*

$$\sup\{-v(x) + xr : x \geq 0\} \leq K \sup\{-v_1(x) + xr : x \geq 0\}$$

*or*

$$\sup\{-v(x) + xr : x \geq 0\} \leq K \sup\{-v_1(x) + xKr : x \geq 0\},$$

*where  $K = \text{const} > 0$ .*

**Proposition 2.** *Let  $0 < R \leq +\infty$ ,  $\Phi \in \Omega(R)$ ,  $\Phi'(r)/\Phi(r)$  be a function, nondecreasing on  $[r_0, R)$ , and*

$$(27) \quad \frac{\Phi''(r)\Phi(r)}{(\Phi'(r))^2} \leq H < +\infty, \quad r \in [r_0, R).$$

*Suppose that  $\varphi$  is a characteristic function on probability law  $F$  analytic in  $\mathbb{D}_R$  such that (8) holds. If  $\ln \mu(r, \varphi)$  belongs to a convergence  $\Phi$ -class then (24) holds.*

*Proof.* Since the function  $\Phi'(r)/\Phi(r)$  is nondecreasing on  $[r_0, R]$ , the function

$$\frac{\Phi'(r) \ln \mu(r, \varphi)}{\Phi(r)}$$

is continuous and increasing to  $+\infty$  on  $[r_0, R]$ . Therefore, there exists a function  $r(x)$  increasing to  $+\infty$  and continuous on  $(x_0, +\infty)$  such that

$$\frac{\Phi'(r(x)) \ln \mu(r(x), \varphi)}{\Phi(r(x))} = x,$$

and since in view of (22)

$$\int_{r_0}^R (l(r)/\Phi(r)) dr < +\infty,$$

we obtain

$$\int_{x_0}^{\infty} \frac{x}{\Phi(r(x))} dx = \int_{x_0}^{\infty} \frac{l(r(x))}{\Phi(r(x))} dx < +\infty.$$

As above, let  $B(r) = \int_r^R \frac{dx}{\Phi(x)}$ . Using the l'Hospital rule we have

$$\lim_{r \rightarrow R} B(r)\Phi(r) = \lim_{r \rightarrow R} \frac{\int_r^R \frac{dx}{\Phi(x)}}{1/\Phi(r)} = \lim_{r \rightarrow R} \frac{\Phi(r)}{\Phi'(r)} < +\infty.$$

Therefore,

$$\begin{aligned} xB(r(x)) &= \frac{\Phi'(r(x)) \ln \mu(r(x), \varphi)}{\Phi(r(x))} = \\ &= \Phi(r(x))B(r(x)) \frac{\Phi'(r(x)) \ln \mu(r(x), \varphi)}{\Phi^2(r(x))} = O(1), \quad x \rightarrow +\infty, \end{aligned}$$

and

$$\int_{x_0}^{\infty} B(r(x)) dx = xB(r(x)) \Big|_{x_0}^{+\infty} - \int_{x_0}^{\infty} x dB(r(x)) = \text{const} + \int_{x_0}^{\infty} \frac{x}{\Phi(r(x))} dx < +\infty.$$

From (27) it follows that

$$B(r) = \int_r^R \frac{dr}{\Phi(r)} \geq \frac{1}{H} \int_r^R \frac{\Phi''(r) dr}{(\Phi'(r))^2} = \frac{1}{H\Phi'(r)}.$$

Therefore,

$$(28) \quad \int_{x_0}^{\infty} \frac{dx}{\Phi'(r(x))} < +\infty.$$

But  $\ln \mu(r(x), \varphi) \geq \ln W_F(x) + xr(x)$ , that is

$$r(x) \leq \frac{\ln \mu(r(x), \varphi)}{x} + \frac{1}{x} \ln \frac{1}{W_F(x)} = \frac{\Phi(r(x))}{\Phi'(r(x))} + \frac{1}{x} \ln \frac{1}{W_F(x)},$$

whence

$$\Psi(r(x)) \leq \frac{1}{x} \ln \frac{1}{W_F(x)}.$$

For some  $\xi = \xi(r)$  we have

$$0 \leq \ln \Phi'(r) - \ln \Phi'(\Psi(r)) = \frac{\Phi''(\xi)}{\Phi'(\xi)}(r - \Psi(r)) = \frac{\Phi''(\xi)}{\Phi'(\xi)} \frac{\Phi(r)}{\Phi'(r)} \leq \frac{\Phi''(\xi)\Phi(\xi)}{(\Phi'(\xi))^2} \leq H$$

i.e.  $\Phi'(r) = O(\Phi'(\Psi(r)))$  as  $r \uparrow R$  and, thus,

$$\Phi'(r(x)) \leq K\Phi'(\Psi(r(x))) \leq K\Phi' \left( \frac{1}{x} \ln \frac{1}{W_F(x)} \right),$$

where  $K = \text{const} > 0$ . Hence and from (28) we obtain (24). Proposition 2 is proved.

*Remark 2.* Let

$$0 < R < +\infty \quad \text{and} \quad \Phi(r) = A \left( \frac{1}{R-r} \right).$$

Then the function  $\Phi'(r)/\Phi(r)$  is nondecreasing on  $[r_0, R)$  if and only if the function  $x^2 A'(x)/A(x)$  is nondecreasing on  $[x_0, +\infty)$ . The latter condition does not influence  $A$  on speed of growth but influences its smoothness. If  $R = +\infty$  the function  $\Phi'(r)/\Phi(r)$  is nondecreasing on  $[r_0, R)$  if  $\Phi$  does not increase slower than the exponential function. For the power functions this condition does not hold. It seems to us that the condition  $\Phi'(r)/\Phi(r) \nearrow$  may be replaced by the condition  $(\Phi'(r))^{1+\eta}/\Phi(r) \nearrow$  for some  $\eta \in [0, 1)$ .

#### 4. Estimates of $\ln I(r, \varphi)$ by $\ln \mu(r, \varphi)$

We suppose that  $\Phi \in \Omega(R)$  and

$$(29) \quad \Phi'(r) > \frac{1}{R-r}, \quad r_0 < r < R.$$

Then  $r + 1/\Phi'(r) < R$ ,

$$\begin{aligned} I(r, \varphi) &= \int_0^\infty W_F(x) e^{xr} dx = \int_0^\infty W_F(x) \exp\{x(r + 1/\Phi'(r))\} \exp\{x/\Phi'(r)\} dx \leq \\ &\leq \mu(r + 1/\Phi'(r), \varphi) \Phi'(r) \end{aligned}$$

and

$$\ln I(r, \varphi) \leq \ln \mu(r + 1/\Phi'(r), \varphi) + \ln \Phi'(r)$$

for all  $r_0 < r < R$ . Therefore, if

$$(30) \quad \int_{r_0}^R \frac{\Phi'(r) \ln \Phi'(r)}{\Phi^2(r)} dr < +\infty,$$

then

$$\int_{r_0}^R \frac{\Phi'(r) \ln I(r, \varphi)}{\Phi^2(r)} dr \leq \int_{r_0}^R \frac{\Phi'(r) \ln \mu(r + 1/\Phi'(r), \varphi)}{\Phi^2(r)} dr + \text{const.}$$

Using this inequality it is easy to prove following proposition.

**Proposition 3.** Let  $0 < R \leq +\infty$ ,  $\Phi \in \Omega(R)$ , the conditions (27), (29), (30) hold and  $\Phi'(r + 1/\Phi'(r)) \leq H_1 \Phi'(r)$  for all  $r \in [r_0, R)$ , where  $H_1 = \text{const} > 0$ . Let  $\varphi$  be a characteristic function on probability law  $F$  analytic in  $\mathbb{D}_R$ . If  $\ln \mu(r, \varphi)$  belongs to a convergence  $\Phi$ -class then  $\ln I(r, \varphi)$  belongs to such class as well.

*Proof.* Since  $\Phi'(r + 1/\Phi'(r)) \leq H_1 \Phi'(r)$ , we have in view of (27) for some  $\xi = \xi(r) \in (r, r + 1/\Phi'(r))$

$$\begin{aligned} & \left| \ln \frac{\Phi^2(r + 1/\Phi'(r))}{\Phi'(r + 1/\Phi'(r))} - \ln \frac{\Phi^2(r)}{\Phi'(r)} \right| = \left| 2 \frac{\Phi'(\xi)}{\Phi(\xi)} - \frac{\Phi''(\xi)}{\Phi'(\xi)} \right| \frac{1}{\Phi'(r)} \\ & = \frac{\Phi'(\xi)}{\Phi(\xi)\Phi'(r)} \left| 2 - \frac{\Phi''(\xi)\Phi(\xi)}{(\Phi'(\xi))^2} \right| \leq H_1 \frac{\Phi'(\xi)}{\Phi(\xi)\Phi'(\xi)} \left| 2 - \frac{\Phi''(\xi)\Phi(\xi)}{(\Phi'(\xi))^2} \right| \rightarrow 0, \quad r \uparrow R \end{aligned}$$

that is

$$\frac{\Phi'(r)}{\Phi^2(r)} = (1 + o(1)) \frac{\Phi'(r + 1/\Phi'(r))}{\Phi^2(r + 1/\Phi'(r))}, \quad r \uparrow R.$$

From (27) it follows also that  $\Phi''(r) = o((\Phi'(r))^2)$  as  $r \uparrow R$ , that is

$$1 - \Phi''(r)/(\Phi'(r))^2 \geq h_1 > 0.$$

Therefore, for some  $r_1 \geq r_0$  in view of (30) we obtain

$$\begin{aligned} & \int_{r_1}^R \frac{\Phi'(r) \ln I(r, \varphi)}{\Phi^2(r)} dr \leq \int_{r_1}^R \frac{\Phi'(r) \ln \mu(r + 1/\Phi'(r), \varphi)}{\Phi^2(r)} dr + \text{const} \\ & \leq 2 \int_{r_1}^R \frac{\Phi'(r + 1/\Phi'(r)) \ln \mu(r + 1/\Phi'(r), \varphi)}{\Phi^2(r + 1/\Phi'(r))} \frac{d(r + 1/\Phi'(r))}{1 - \Phi''(r)/(\Phi'(r))^2} + \text{const} \\ & \leq \frac{2}{h_1} \int_{r_1}^R \frac{\Phi'(r + 1/\Phi'(r)) \ln \mu(r + 1/\Phi'(r), \varphi)}{\Phi^2(r + 1/\Phi'(r))} d(r + 1/\Phi'(r)) + \text{const} < +\infty. \end{aligned}$$

Proposition 3 is proved.

*Remark 3.* The condition (30) is significant. In [9] it is shown that this condition is near to necessary in order that the logarithms of the maximum modulus of an entire function and the maximal term of its power development belong to the same convergence  $\Phi$ -class.

For  $R = +\infty$  the inequality (29) is trivial. Clearly in the case  $R < +\infty$  the condition (29) is natural.

Finally, the condition  $\Phi'(r+1/\Phi'(r)) \leq H_1 \Phi'(r)$  for  $r \in [r_0, R)$  is a condition on the smoothness of  $\Phi'$ . If  $R = +\infty$  then by Borel-Nevanlinna theorem [2, p. 120–121]  $\Phi'(r+1/\Phi'(r)) \leq (1+\varepsilon)\Phi'(r)$  for every  $\varepsilon > 0$  and all  $r \geq r_0$  outside of a set of finite measure.

### 5. Main result and colloraries

Hence we obtain a theorem on the belonging of the characteristic function  $\varphi$  of a probability law  $F$  analytic in  $\mathbb{D}_R$  to convergence  $\Phi$ -class, that is

$$\int_{r_0}^R \frac{\Phi'(r) \ln M(r, \varphi)}{\Phi^2(r)} dr < +\infty.$$

At first we remark that if  $\Phi'(r)/\Phi(r)$  be a function, nondecreasing on  $[r_0, R)$  then  $\Phi''(r)\Phi(r) - (\Phi'(r))^2 \geq 0$  and, thus, the condition (23) holds. Therefore, using the inequalities (7) from Propositions 1–3 we obtain the following main result:

**Theorem.** *Let  $0 < R \leq +\infty$ ,  $\Phi \in \Omega(R)$ ,  $\Phi'(r)/\Phi(r)$  be a function, nondecreasing on  $[r_0, R)$ ,  $\Phi'(r+1/\Phi'(r)) \leq H_1 \Phi'(r)$  for all  $r \in [r_0, R)$ , where  $H_1 = \text{const} > 0$ , and the condition (27) (29), (30) hold. Suppose that  $\varphi$  is an analytic in  $\mathbb{D}_R$  characteristic function on probability law  $F$  such that (8) holds.*

*Then in order that  $\varphi$  belongs to a convergence  $\Phi$ -class it is necessary and in the case, when*

$$\ln \frac{1}{W_F(x)} = v(x) \in V(R),$$

*it is sufficient that (24) holds.*

We bring several colloraries to this theorem. At first we suppose that

$$0 < R < \infty, \quad 0 < \varrho < \infty \Phi(r) = (R-r)^{-\varrho}.$$

Then

$$\Phi'(r) = \varrho(R-r)^{-(\varrho+1)} > (R-r)^{-1}, \quad \frac{\Phi'(r)}{\Phi(r)} = \frac{1}{\varrho(R-r)} \uparrow +\infty \quad \text{as } r \uparrow R,$$

$$\Phi'(r) > (R-r)^{-1}, \quad \Phi'(r+1/\Phi'(r)) = (1+o(1))\Phi'(r) \quad \text{as } r \uparrow R$$

and

$$\Phi(r)\Phi''(r)(\Phi(r))^{-2} = \frac{\varrho+1}{\varrho}.$$

Thus, the fuction  $\Phi(r) = (R-r)^{-\varrho}$  satisfies all conditions of the Theorem. We remark also that

$$\Phi' \left( \frac{1}{x} \ln \frac{1}{W_F(x)} \right) = \varrho \left( R - \frac{1}{x} \ln \frac{1}{W_F(x)} \right)^{-(\varrho+1)} = \left( \frac{\ln(W_F(x)e^{Rx})}{x} \right)^{-(\varrho+1)}.$$

Therefore, our Theorem implies the following collorary.

**Corollary 2.** Let  $0 < R < +\infty$ ,  $0 < \varrho < \infty$  and  $\varphi$  be a characteristic function on probability law  $F$  analytic in  $\mathbb{D}_R$ , such that (8) holds. Then in order that

$$\int_{r_0}^R (R-r)^{\varrho-1} \ln M(r, \varphi) dr < +\infty,$$

it is necessary, and in the case when

$$\ln \frac{1}{W_F(x)} = v(x) \in V(R),$$

it is sufficient that

$$\int_{x_0}^{\infty} \left( \frac{\ln(W_F(x)e^{Rx})}{x} \right)^{\varrho+1} dx < +\infty.$$

Now we suppose that  $0 < \varrho < \infty$  and

$$(31) \quad \Phi(r) = \frac{1}{(R-r)^2} \exp \left\{ \frac{\varrho}{R-r} \right\}.$$

Then

$$\begin{aligned} \Phi'(r) &= \left( \frac{2}{(R-r)^3} + \frac{\varrho}{(R-r)^4} \right) \exp \left\{ \frac{\varrho}{R-r} \right\} = \\ &= \frac{(1+o(1))\varrho}{(R-r)^4} \exp \left\{ \frac{\varrho}{R-r} \right\}, \quad r \uparrow R, \\ \Phi''(r) &= \frac{(1+o(1))\varrho^2}{(R-r)^6} \exp \left\{ \frac{\varrho}{R-r} \right\}, \quad r \uparrow R, \end{aligned}$$

and, therefore,

$$\frac{\Phi'(r)}{\Phi(r)} = \frac{2}{R-r} + \frac{\varrho}{(R-r)^2} \uparrow +\infty, \quad \Phi'(r + 1/\Phi'(r)) = (1+o(1))\Phi'(r)$$

and

$$\frac{\Phi''(r)\Phi(r)}{(\Phi'(r))^2} = 1 + o(1),$$

as  $r \uparrow R$ . Thus, the function (31) satisfies all conditions of the Theorem and we obtain the following

**Corollary 3.** Let  $0 < R < +\infty$ ,  $0 < \varrho < \infty$  and  $\varphi$  be a characteristic function on probability law  $F$  analytic in  $\mathbb{D}_R$ , such that (8) holds. Then in order that

$$\int_{r_0}^R \varrho \exp \left\{ -\frac{\varrho}{R-r} \right\} \ln M(r, \varphi) dr < +\infty,$$

it is necessary, and in the case when

$$\ln \frac{1}{W_F(x)} = v(x) \in V(R),$$

it is sufficient that

$$\int_{x_0}^{\infty} \exp \left\{ -\frac{\varrho x}{\ln(W_F(x)e^{Rx})} \right\} \left( \frac{\ln(W_F(x)e^{Rx})}{x} \right)^4 dx < +\infty.$$

Finally, let

$$R = +\infty, \quad 0 < \varrho < \infty \quad \text{and} \quad \Phi(r) = e^{\varrho r}.$$

Then

$$\Phi'(r) = \varrho e^{\varrho r}, \quad \Phi'(r)/\Phi(r) = 1/\varrho, \quad \Phi(r)\Phi''(r)(\Phi(r))^{-2} = 1$$

and

$$\Phi'(r + 1/\Phi'(r)) = (1 + o(1))\Phi'(r) \quad \text{as} \quad r \rightarrow +\infty,$$

that is all conditions of the Theorem hold, and we obtain the following

**Corollary 4.** Let  $0 < \varrho < \infty$  and  $\varphi$  be an entire characteristic function on probability law  $F$ . Then in order that

$$\int_0^{\infty} e^{-\varrho r} \ln M(r, \varphi) dr < +\infty,$$

it is necessary and, in the case when

$$\ln \frac{1}{W_F(x)} = v(x) \in V(+\infty),$$

it is sufficient that

$$\int_0^{\infty} (W_F(x))^{\varrho/x} dx < +\infty.$$

## References

- [1] Ju. V. Linnik and I. V. Ostrovskii, *Decomposition of random variables and vectors*, Moscow: Nauka (1972), 479, in Russian.
- [2] A. A. Gol'dberg and I. V. Ostrovskii, *Value distribution of meromorphic functions*, Moscow: Nauka (1970), 592, in Russian.
- [3] Ju. M. Gal' and M. N. Sheremeta, *Belonging of analytic functions to a convergence class*, *Dopovidi AN URSR*, Ser. A 7 (1985), 11-14, in Ukrainian.
- [4] G. R. MacLane, *Asymptotic values of holomorphic function*, *Rice Univ. Studies* 49 (1963), 83.
- [5] M. N. Sheremeta, *On belonging of entire characteristic functions on probability laws to a convergence class*, *Dopovidi AN URSR*, Ser. A 8 (1975), 696-699, in Ukrainian.
- [6] V. M. Sorokivskii, *On the growth of characteristic functions on probability laws*, *Dopovybych* 1980, 20. Manuscript dep. in VINITI 17.12.80, N 5330-80 Dep., in Russian.

- [7] M. N. Sheremeta and S. I. Fedynyak, *On the derivative of Dirichlet series*, Siberian Math. J. **39**, no. 1 (1998), 206–223.
- [8] O. M. Mulyava and M. M. Sheremeta, *On a convergence class for Dirichlet series*, Bull. Soc. Lettres Łódź **50** Ser. Rech. Deform. **30** (2000), 23–30.
- [9] P. V. Filevych and M. M. Sheremeta, *On a convergence class for entire functions*, Bull. Soc. Lettres Łódź **53** Ser. Rech. Deform. **40** (2003), 5–16.
- [10] O. M. Mulyava, M. M. Sheremeta, and O. M. Sumyk, *Relation between the maximum modulus and the maximal term of Dirichlet series in terms of a convergence class*, Bull. Soc. Lettres Łódź, Ser. Rech. Deform. **62**, no. 2 (2012), 93–105.

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## O PRZNALEŻNOŚCI FUNKCJI CHARAKTERYSTYCZNYCH PRAW PRAWDOPODOBIENSTWA DO KLASY ZBIEŻNOŚCI

### Streszczenie

Badany jest warunek na prawo prawdopodobieństwa, przy którym jego funkcja charakterystyczna należy do pewnej klasy zbieżności.