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ORIGINAL PAPER

Axisymmetric problem for a spherical crack on the interface of elastic media

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Abstract A problem concerning a spherical interfacial crack is solved by the eigenfunction method. The problem is reduced to a coupled system of dual-series equations in terms of Legendre functions and then to a system of singular integral equations for two unknown functions. The behaviour of the solution near the edge of the spherical crack, and the stress-intensity factors and crack-opening displacements are studied. The case when the crack surfaces are under normal internal pressure of constant intensity is examined.

Keywords Cavity · Composite · Elastic · Inclusion · Interface spherical crack

1 Introduction

Aging and damage of materials are processes that are the causes of many dramatic events in the world. They may lead to catastrophic failures in oil and gas-storage tanks, pressure vessels, turbine-generator rotors, steam boilers, pipelines, bridges, airplanes, railways and welded ships [1]. It is also known that electronic chips sometimes become disfunctional due to mechanical damage. Scientific and engineering evidence shows that cracks in materials are the first steps in a sequence of processes leading to their fracture. Internal cracks, in the form of breaks in material solidity, have been examined in the literature for quite a long time [1, 2]. Special attention [3, 4] has been paid to linear crack problems in unbounded elastic bodies; the main results were obtained for flat penny-shaped or elliptic cracks. A comprehensive review of the state-of-the-art can be found in [2, 5, 6]. However, according to experimental analysis of the surfaces of damaged objects, the initial surfaces of the material breaks are not flat, being mainly of spherical or ellipsoidal shape [5, 7]. To evaluate the strength of a material with internal cracks, one can start from the solution of a class of problems within the theory of elasticity for three-dimensional bodies weakened by cracks with curved surfaces. Such cracks could be modelled by cuts on a part of some surface of revolution having non-zero curvature. In this case, one has the possibility to vary the geometrical parameters of the

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$$\Psi^+(x) = \frac{1}{2}f(x) + \frac{1}{4\pi \mathrm{i}}\int_{-\theta_0}^{\theta_0} f(t)\mathrm{cosec}\frac{t-x}{2}\mathrm{d}t,$$

$$\Psi^-(x) = -\frac{1}{2}f(x) + \frac{1}{4\pi \mathrm{i}}\int_{-\theta_0}^{\theta_0} f(t)\mathrm{cosec}\frac{t-x}{2}\mathrm{d}t, \tag{29}$$

$$G = \frac{1-2\gamma}{1+2\gamma}, \quad g(t) = \frac{F(t)}{1+2\gamma}.$$

Having solved the Riemann problem, we can represent the characteristic equation as

$$f(x) = \frac{Z(x)}{1-4\gamma^2}\left[F(x) - \frac{\gamma}{\pi \mathrm{i}}\int_{-\theta_0}^{\theta_0}\frac{F(t)-F(x)}{Z(t)\sin\frac{t-x}{2}}\mathrm{d}t\right], \tag{30}$$

where

$$Z(x) = \sqrt{1-4\gamma^2}\left(\tan\frac{\theta_0 - x}{4}\right)^{\mathrm{i}\lambda}\left(\tan\frac{\theta_0+x}{4}\right)^{-\mathrm{i}\lambda}, \quad \lambda = \frac{1}{2\pi}\log\frac{1+2\gamma}{1-2\gamma}. \tag{31}$$

This solution of the characteristic equation (26) allows us to convert the singular equation (24) into a Fredholm equation, similar to the regularization of the singular equations as described in [24].

Let us rewrite Eq. (24) as follows

$$f(x) + \frac{\gamma}{\pi \mathrm{i}}\int_{-\theta_0}^{\theta_0}\frac{f(t)\,\mathrm{d}t}{\sin\frac{t-x}{2}} = \Phi(x) - kf(x). \tag{32}$$

We temporarily assume that the right part of Eq. (32) is a known function. It is easy to show that the function $kf(x)$ is regular. Solving Eq. (32) by applying (21), we get

$$f(x) + \frac{Z(x)}{1-4\gamma^2}\left[kf(x) - \frac{\gamma}{\pi \mathrm{i}}\int_{-\theta_0}^{\theta_0}\frac{kf(t)-kf(x)}{Z(t)\sin\frac{t-x}{2}}\mathrm{d}t\right] = \frac{Z(x)}{1-4\gamma^2}\left[\Phi(x) - \frac{\gamma}{\pi \mathrm{i}}\int_{-\theta_0}^{\theta_0}\frac{\Phi(t)-\Phi(x)}{Z(t)\sin\frac{t-x}{2}}\mathrm{d}t\right]. \tag{33}$$

Thus, the singularity of the solution of Eq. (24) at the end of the interval $[-\theta_0, \theta_0]$ is determined by the function $Z(t)$. Hence, $f(t)$ can be represented in the form

$$f(t) = Z(t)\,L(t), \quad L(t) = L_1(t) + \mathrm{i}L_2(t), \quad L(-t) = \bar{L}(t), \tag{34}$$

where $L(t)$ does not have singularities at the ends of the interval $[-\theta_0, \theta_0]$.

By introducing

$$\frac{1}{1-4\gamma^2}\left[\Phi(x) - \frac{\gamma}{\pi \mathrm{i}}\int_{-\theta_0}^{\theta_0}\frac{\Phi(t)-\Phi(x)}{Z(t)\sin\frac{t-x}{2}}\mathrm{d}t\right] = F_*(x), \tag{35}$$

we may easily show that Eq. (33) is equivalent to the Fredholm system

$$L(x) + \frac{1}{1-4\gamma^2}\int_{-\theta_0}^{\theta_0}\left[L(t)\,M_1(t,x) + \bar{L}(t)\,M_2(t,x)\right]\mathrm{d}t = F_*(x), \tag{36}$$

$$M_1(t,x) = \frac{\mathrm{i}\gamma Z(t)}{\pi\left(1-4\gamma^2\right)}\int_{-\theta_0}^{\theta_0}\bar{Z}(\tau)\frac{K_1(t,\tau) - K_1(t,x)}{\sin\frac{\tau - x}{2}}\mathrm{d}\tau + Z(t)\,K_1(t,x), \tag{37}$$

$$M_2(t,x) = \frac{\mathrm{i}\gamma \bar{Z}(t)}{\pi\left(1-4\gamma^2\right)} \int_{-\theta_0}^{\theta_0} \bar{Z}(\tau) \frac{K_2(t,\tau) - K_2(t,x)}{\sin\frac{\tau - x}{2}} \mathrm{d}\tau + \bar{Z}(t) K_2(t,x).$$

The kind and properties of the functions $Z(t)$ and $L(t)$ allow us to separate real and imaginary parts in Eq. (36) and to obtain a system of two Fredholm integral equations for the functions $L_1(t)$ and $L_2(t)$ on the interval $[0, \theta_0]$.

4 Results and discussions

Let us investigate the stress and displacement fields near the edge of the spherical cut. It is known that, while calculating these characteristics near the contour of the penny-shaped crack placed on the boundary of the division of two different materials, mathematical difficulties in estimating asymptotically the number of integrals arise. It can be proved that near the boundary circumference of the spherical cut, that is, for $\theta \simeq \theta_0 - \varepsilon$, $(\varepsilon \ll 1)$, the difference of the displacement in the first approximation is represented by the integral

$$\Delta u_\theta + \mathrm{i}\Delta u_r \approx \frac{2r_0}{G_1 G_2} \int_\theta^{\theta_0} \frac{Z(t)\, L(t)\, \mathrm{d}t}{\sqrt{2\cos\theta - 2\cos t}} = \frac{2r_0\sqrt{1-4\gamma^2}}{G_1 G_2} \int_\theta^{\theta_0} \left(\frac{\tan\frac{\theta_0 - t}{4}}{\tan\frac{\theta_0 + t}{4}} \right)^{\mathrm{i}\lambda} \frac{L(t)\, \mathrm{d}t}{\sqrt{2\cos\theta - 2\cos t}}, \tag{38}$$

where

$$\Delta u_r = u_r^{(1)} - u_r^{(2)} = \frac{r_0}{G_1 G_2} \sum_{n=0}^{\infty} I_n^{(1)} P_n(\cos\theta),$$

$$\Delta u_\theta = u_\theta^{(1)} - u_\theta^{(2)} = \frac{r_0}{G_1 G_2} \sum_{n=0}^{\infty} I_n^{(2)} \frac{n + 1/2}{n(n+1)} P_n^{(1)}(\cos\theta).$$

Applying the method of asymptotic integration to the integral (38), where $\theta \sim \theta_0$, we get

$$\Delta u_\theta + \mathrm{i}\Delta u_r \approx \frac{2r_0 L(\theta_0)\sqrt{1-4\gamma^2}}{\cos\frac{1}{2}\theta_0} \left(\frac{\tan\frac{\theta_0 - \theta}{4}}{\tan\frac{\theta_0 + \theta}{4}} \right)^{\frac{1}{2} + \mathrm{i}\lambda} \cdot \frac{\Gamma\left(\frac{1}{2}\right)\Gamma(1 + \mathrm{i}\lambda)}{\Gamma\left(\frac{3}{2} + \mathrm{i}\lambda\right)}, \tag{39}$$

where $\Gamma()$ is the Gamma function. It should be noted that (39) can be derived from (38) by different methods. But, obviously, the most rational one is the substitution

$$\sin\frac{t}{2} = \sin\frac{\theta}{2} + S\left(\sin\frac{\theta_0}{2} - \sin\frac{\theta}{2}\right). \tag{40}$$

The stress fields on the surface of the sphere outside the cut are defined by the right parts of the equalities (12) when $\theta > \theta_0$. An analysis of the series (12) proves that the main stress components for $\theta \approx \theta_0 + \varepsilon$ $(\varepsilon \ll 1)$ are found from the sums

$$\sigma_r = \sum_{n=0}^{\infty} \left[a_0 I_n^{(1)} + b_0 I_n^{(2)} \right] \left(n + \frac{1}{2} \right) P_n(\cos\theta),$$

$$\sigma_{r\theta} = \sum_{n=1}^{\infty} \left[\frac{1}{2} b_0 I_n^{(1)} + 2a_0 I_n^{(2)} \right] \frac{\left(n + \frac{1}{2}\right)^2 P_n^1(\cos\theta)}{n(n+1)}. \tag{41}$$

Considering correlations for the Legendre functions [23], asymptotic expressions for the stress fields can be represented in the form

$$\sigma_r \approx -\frac{1}{\sin\theta}\frac{\mathrm{d}}{\mathrm{d}\theta}\left\{\sin\theta\sum_{n=1}^{\infty}\left[a_0 I_n^{(1)} + b_0 I_n^{(2)}\right]\frac{\left(n+\frac{1}{2}\right)P_n^1(\cos\theta)}{n(n+1)}\right\};$$
$$\sigma_{r\theta} \approx \frac{\mathrm{d}}{\mathrm{d}\theta}\sum_{n=0}^{\infty}\left[\frac{1}{2}b_0 I_n^{(1)} + 2a_0 I_n^{(2)}\right]P_n(\cos\theta), \quad (\theta > \theta_0). \tag{42}$$

If $I_n^{(1)}, I_n^{(2)}$ in (42) are substituted by the integral operators (8) and the order of summation and integration is changed, we have

$$\sigma_r \approx \frac{2a_0}{\sin\theta}\frac{\mathrm{d}}{\mathrm{d}\theta}\int_\theta^{\theta_0}\frac{\sin t\varphi(t)\,\mathrm{d}t}{\sqrt{2\cos t - 2\cos\theta}}, \quad \sigma_{r\theta} \approx 2a_0\frac{\mathrm{d}}{\mathrm{d}\theta}\int_\theta^{\theta_0}\frac{\psi(t)\,\mathrm{d}t}{\sqrt{2\cos t - 2\cos\theta}}. \tag{43}$$

On the basis of (25), (31), (34) the last relationships, where $\theta \sim \theta_0$, can be presented in the following equivalent form

$$\sigma_{r\theta}+\mathrm{i}\sigma_r \approx -2a_0\sin\theta_0 L(\theta_0)\sqrt{1-4\gamma^2}\int_\theta^{\theta_0}\left(\frac{\tan\frac{\theta_0-t}{4}}{\tan\frac{\theta_0+t}{4}}\right)^{\mathrm{i}\lambda}\frac{\mathrm{d}t}{\sqrt{2\cos t - 2\cos\theta}}. \tag{44}$$

After asymptotic integration we will get

$$\sigma_{r\theta} + \mathrm{i}\sigma_r \approx (K_2 + \mathrm{i}K_1)\,r_0^{-0.5}\left(\sin\frac{\theta}{2} - \sin\frac{\theta_0}{2}\right)^{-\frac{1}{2}+\mathrm{i}\lambda}\cdot\left(\sin\frac{\theta}{2} + \sin\frac{\theta_0}{2}\right)^{-\frac{1}{2}-\mathrm{i}\lambda}, \tag{45}$$

where the normal K_1 and tangential K_2 SIF are determined from

$$K_2 + \mathrm{i}K_1 = -4a_0\sqrt{1-4\gamma^2}\sqrt{r_0}L(\theta_0)\cdot\frac{\Gamma(1+\mathrm{i}\lambda)\,\Gamma\left(\frac{1}{2}-\mathrm{i}\lambda\right)}{\Gamma\left(\frac{3}{2}\right)}. \tag{46}$$

The formulas for K_1 and K_2 could be reduced to the results for a penny-shaped crack between dissimilar half-spaces (the problem was studied by Willis [27]) in the limit $r_0 \to \infty$. One of the authors of this paper has carried out such an operation for penny-shaped and spherical cracks in homogeneous space [28]. This demonstrates the reliability of the method presented here. The character of the asymptotic stress field near the boundary circumference of the spherical cut on the interface boundary is the same as near the edge of a penny-shaped crack between two dissimilar half-spaces. If the oscillatory character in this problem is determined by the multipliers $(\sin\theta/2 - \sin\theta_0/2)^{-1/2+\mathrm{i}\lambda}$, then, near the penny-shaped crack, it is determined by the multipliers $(r - r_0)^{-1/2+\mathrm{i}\lambda}$.

We shall now consider the case of an external uniform expansion when the cut surfaces are loaded by normal internal pressure of intensity q. The boundary conditions (2) have the following form

$$\sigma_r^{(1)} = \sigma_r^{(2)}, \quad \sigma_{r\theta}^{(1)} = \sigma_{r\theta}^{(2)}, \quad u_r^{(1)} = u_r^{(2)}, \quad u_\theta^{(1)} = u_\theta^{(2)}, (r = r_0, \theta_0 \le \theta \le \pi),$$
$$\sigma_r^{(1)} = \sigma_r^{(2)} = f_1(\theta) = -q, \quad \sigma_{r\theta}^{(1)} = \sigma_{r\theta}^{(2)} = f_2(\theta) = 0, \quad (r = r_0, 0 \le \theta < \theta_0), \tag{47}$$

The right parts of system (36) are easy to solve. Note that the solution of the system of integral equations (36) depends on the elastic characteristics of the material ν_1, ν_2, G_1, G_2, geometry of the cut r_0, θ_0 and the condition of loading on the surface of the cut. Each of the mentioned parameters has a rather wide range of variation. In this article we will provide conclusions and numerical results which could be of practical value.

First of all, it is worth noting that the dependence of the SIF on Poisson's ratio, having their values in the interval $0{\cdot}2 < \nu_1, \nu_2 < 0{\cdot}42$, is insignificant. For example, with $\nu_1 = \nu_2 = 1/3$ and $\nu_1 = 1/3, \nu_2 = 1/4$ $(G_1 = G_2)$ SIF K_1 and K_2 are, respectively, equal to 0.457 and 0.462.

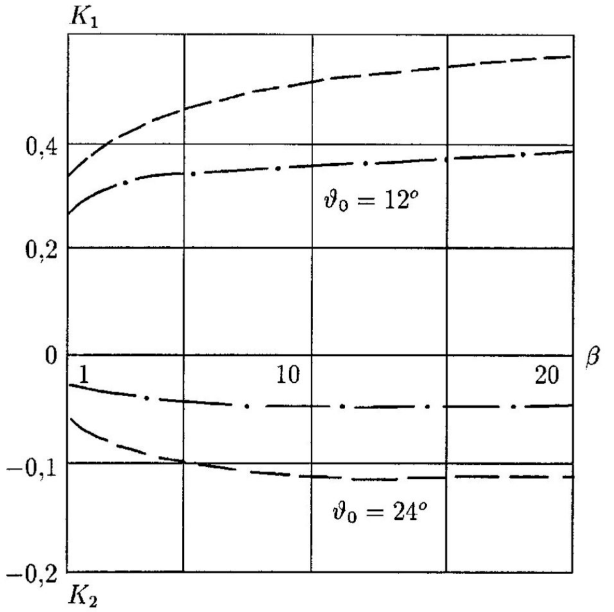

Fig. 3 Dependence of normal K_1 and tangential K_2 SIF on the ratio of shear modules $\beta = G_1/G_2$ inclusion and matrix for different values of the angle of the cut ($\theta_0 = 12; 24$)

Figure 3 shows the SIF behaviour depending on the ratio of shear moduli $\beta = G_1/G_2$ inclusion and matrix for different values of the angle of the cut ($\theta_0 = 12; 24$). Obviously, the main changes of SIF K_1 and K_2 take place when β increases to values of $\beta \gg 5$. This means that a further increase of the ratio of shear moduli of inclusion and matrix should not lead to significant changes in the behaviour of the crack on the interface in composite materials.

These results allow us to compare SIF for composite ($\beta > 1$) and homogeneous ($\beta = 1$) materials. The case $0 < \beta < 1$ is not of practical interest. Figure 3 shows that, with for increased values of β, SIF increase and exceed the corresponding values of SIF for a homogeneous material. If we introduce the parameters $S_1 = K_1^{(c)}/K_1^{(0)}, S_2 = K_2^{(c)}/K_2^{(0)}$, (upper indices correspond to composite and homogeneous materials), then, with $1 < \beta < 60$, the parameters S_i ($i = 1, 2$) are in the interval $1 < S_i < 1{\cdot}5$.

5 Concluding remarks

The class of problems linked to the interaction of matrix and inclusion is interesting from the point of view of the mechanics of composite materials reinforced by hard particles. Cracks on the interface boundary of matrix and inclusion in such materials could appear due to mechanical coercion or environmental influence. Several scientific works present mostly numerical methods for the solution of this class of problems, which is explained by mathematical difficulties arising in the analytical approach. However, only an analytical solution is capable of covering all the singularities of the problem, and giving a general picture of the mechanical conditions of the system dependency on changes in the problem parameters, such as external loading, geometry of the crack, elastic constants of matrix and inclusion, etc., and, thus, to predict the behaviour of the cut when these parameters change. The results of this work show the advantages of such an approach. In particular, we obtained analytical expressions for the components of the stress tensor and the SIF near the edge of the spherical crack on the interface boundary. When the surfaces of the crack are under a normal internal pressure of constant intensity, the dependencies of the SIF on the ratio of the shear moduli β inclusion and the matrix are shown. These dependencies demonstrate that the main changes of the crack's behaviour take place for $\beta \in [1, 5]$. We also compared the values of the SIF for composite and homogeneous materials.

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