On hidden symmetries and solutions of the nonlinear d’Alembert equation

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Abstract
Non-Lie symmetries of nonlinear d’Alembert equation in the pseudo-Euclidean space \( \mathbb{R}_{2,2} \) are studied and new classes of exact solutions are constructed.

1. Introduction
Nonlinear d’Alembert equation is among the most important equations of mathematical and theoretical physics. There are a number of articles that dedicate to obtaining exact solutions of the d’Alembert equations (see, e.g. review paper [16] and references therein). In the present paper we consider a nonlinear d’Alembert equation in the pseudo-Euclidean space \( \mathbb{R}_{2,2} \):

\[
\square u + \lambda u^k = 0,
\]

where \( u = u(x), x = (x_1, x_2, x_3, x_4) \), \( \square u = u_{11} + u_{22} - u_{33} - u_{44}, u_{\alpha\beta} = \partial^2 u / \partial x^\alpha \partial x^\beta (\alpha, \beta = 1, 2, 3, 4) \). It is known [7] that the Eq. (1) is invariant under the extended Poincaré algebra \( AP(2, 2) \) the base of which is generated by the following operators:

\[
P_\alpha = \partial_\alpha, \quad J_{\alpha\beta} = g^{\alpha\gamma} x_\gamma \partial_\beta - g^{\beta\gamma} x_\gamma \partial_\alpha, \quad D = -x^\alpha \partial_\alpha + \frac{2}{k-1} u \partial_u,
\]

where \( \partial_\beta = \partial / \partial x^\beta, \partial_u = \partial / \partial u \), \( g_{11} = g_{22} = -g_{33} = -g_{44} = 1, g_{\alpha\beta} = 0 \) for \( \alpha \neq \beta (\alpha, \beta, \gamma = 1, 2, 3, 4) \).

In the case \( k = 3 \) the maximal invariance algebra of the Eq. (1) is the conformal algebra \( AC(2, 2) = \langle P_\alpha, J_{\alpha\beta}, D, K_\alpha | \alpha, \beta = 1, 2, 3, 4 \rangle \), where the operators \( P_\alpha, J_{\alpha\beta}, D \) are defined by the formulae (2) for \( k = 3 \) and

\[
K_\alpha = -2g_{23}x_\alpha D - x^2 \partial_\alpha, \quad \tilde{x}^2 = x_1^2 + x_2^2 - x_3^2 - x_4^2,
\]

are inversion operators. The searching of the invariant solutions of the Eq. (1) is closely connected with the problem of the classification of subalgebras of algebras \( AP(2, 2) \) and \( AC(2, 2) \). The method of the classification of subalgebras of finite-dimensional algebras Lie was developed in the papers [10–12]. On the basis of this method in [3] the full classification of subalgebras of the conformal algebra \( AC(2, 2) \) was carried out. In [5,6] by using rank 3 subalgebras of the algebra \( AP(2, 2) \) the
symmetry ansatzes which reduce the Eq. (1) to ordinary differential equations were constructed. The rank 3 subalgebras of the algebra AC(2, 2) were used in [2] for constructing solutions of the conformally invariant Eq. (1). The solutions of the Eq. (1) presented in [6] were generalized in [17] on the basis of the method proposed in [4].

The goal of this article is the investigation of the non-Lie symmetries of the Eq. (1) by means of its reduction to two-dimensional equations the symmetries of which are different from the symmetry of the Eq. (1). These symmetries are used for constructing wide classes of exact solutions of the Eq. (1) which include arbitrary functions.

2. On hidden symmetries of the d’Alembert equation

We consider two ansatzes

\[ u = u(y_1, y_2), \quad y_1 = x_1 - x_4, \quad y_2 = x^2_1 + x^2_2 - x^2_4, \]  
\[ u = u(y_1, y_2), \quad y_1 = x_2 - x_3, \quad y_2 = x^2_2 - x^2_3. \]  

The first of them reduces Eq. (1) to the equation

\[ 4y_1u_{12} + 4y_2u_{22} + 8u_2 + \lambda u^k = 0, \]  

and the second one to the equation

\[ 4y_1u_{12} + 4y_2u_{22} + 6u_2 + \lambda u^k = 0, \]  

where \( u_3 = \frac{\partial u}{\partial y_3}, u_{12} = \frac{\partial^2 u}{\partial y_1 \partial y_2} \) (\( x = 1, 2 \)). Evidently, each solution of Eq. (6) and (7) is also a solution of Eq. (1). The Eqs. (6) and (7) are similar and they are different only the coefficients of \( u_2 \), and therefore it should be expected that their group properties identical. It makes the necessary to carry out the investigation the group properties of more general differential equation

\[ 4y_1u_{12} + 4y_2u_{22} + 2nu_2 + \lambda u^k = 0, \]  

where \( u = u(y_1, y_2), n \) is integer number (\( n \geq 3 \)).

**Theorem 1.** The maximal invariance algebra of Eq. (8) for \( k \neq \frac{n}{2} \) and \( n > 2 \) is the four-dimensional Lie algebra the base of which is generated by the following operators:

\[ X_1 = y_1 \frac{\partial}{\partial y_1} + y_2 \frac{\partial}{\partial y_2} - \frac{1}{k - 1} u \frac{\partial}{\partial u}, \quad X_2 = y_2 \frac{\partial}{\partial y_2} - \frac{1}{k - 1} u \frac{\partial}{\partial u}, \]
\[ X_3 = y_1 \frac{\partial}{\partial y_1}, \quad M = y_1^l \left( y_1 \frac{\partial}{\partial y_1} + y_2 \frac{\partial}{\partial y_2} - \frac{n - 2}{2} u \frac{\partial}{\partial u} \right) \quad \text{for} \quad n = 3, \]
\[ M = y_1^{k-2} \left( y_1 \frac{\partial}{\partial y_1} + y_2 \frac{\partial}{\partial y_2} - u \frac{\partial}{\partial u} \right) \quad \text{for} \quad n = 4. \]

are not symmetry operators of Eq. (1), and in literature are known as hidden symmetries (see, e.g. [1] and references therein). Therefore with the help of the ansatzes (4,5) we obtain the reduced Eqs. (6, 7) the symmetries of which are essentially different from the symmetry of Eq. (1).

**Theorem 2.** The maximal invariance algebra of Eq. (8) for \( k = \frac{n}{2} \) and \( n > 2 \) is the four-dimensional Lie algebra the base of which is generated by the following operators:

\[ Z_1 = y_1 \frac{\partial}{\partial y_1} + y_2 \frac{\partial}{\partial y_2} - \frac{n - 2}{2} u \frac{\partial}{\partial u}, \quad Z_2 = y_2 \frac{\partial}{\partial y_2} - \frac{n - 2}{2} u \frac{\partial}{\partial u}, \]
\[ Z_3 = y_1 \frac{\partial}{\partial y_1}, \quad S = y_1 \ln y_1 \frac{\partial}{\partial y_1} + y_2 \ln y_1 \frac{\partial}{\partial y_2} - \frac{n - 2}{2} (\ln y_1 + 1) u \frac{\partial}{\partial u}. \]

Notice that the operators

\[ S = y_1 \ln y_1 \frac{\partial}{\partial y_1} + y_2 \ln y_1 \frac{\partial}{\partial y_2} - \frac{1}{2} (\ln y_1 + 1) u \frac{\partial}{\partial u} \quad \text{for} \quad n = 3, \]
\[ S = y_1 \ln y_1 \frac{\partial}{\partial y_1} + y_2 \ln y_1 \frac{\partial}{\partial y_2} - (\ln y_1 + 1) u \frac{\partial}{\partial u} \quad \text{for} \quad n = 4. \]

are not symmetry operators of Eq. (1).
Denote by $A(n,k)$ the algebra (9). The basis elements of this algebra satisfy the following commutative relations (here and below we indicated only nonzero relations)

$$[X_1, M] = IM, \quad [X_2, X_3] = -X_3.$$  

Our task is the reduction of Eq. (8) to ordinary differential equations. To this end we shall carry out a classification of one-dimensional subalgebras of the algebra $A(n,k)$ up to a contingency with respect to the group of inner automorphisms which is generated by the automorphisms $\text{Ad}(\exp(\alpha X))$, where $X \in A(n,k)$. Using the method of the classification of subalgebras of finite-dimensional Lie algebras stated in [10–12] we receive the following theorem.

**Theorem 3.** Up to the group of inner automorphisms the algebra $A(n,k)$ has the following one-dimensional subalgebras:

1. $K_1 = \langle X_1 + \alpha X_2 \rangle$ ($\alpha \in \mathbb{R}$);
2. $K_2 = \langle X_2 \rangle$;
3. $K_3 = \langle X_1 + \alpha X_3 \rangle$ ($\alpha = \pm 1$);
4. $K_4 = \langle M + \delta X_2 \rangle$ ($\delta = 0, \pm 1$);
5. $K_5 = \langle M + \alpha X_3 \rangle$ ($\alpha = \pm 1$);
6. $K_6 = \langle X_3 \rangle$.

Denote by $A(n)$ the algebra (12). The basis elements of this algebra satisfy the following commutative relations


**Theorem 4.** Up to the group of inner automorphisms the algebra $A(n)$ has the following one-dimensional subalgebras:

1. $L_1 = \langle Z_1 + \delta Z_2 \rangle$ ($\delta = 0, \pm 1$);
2. $L_2 = \langle Z_2 \rangle$;
3. $L_3 = \langle Z_1 + \delta Z_3 \rangle$ ($\delta = \pm 1$);
4. $L_4 = \langle S + \alpha Z_2 \rangle$ ($\alpha \in \mathbb{R}$);
5. $L_5 = \langle S + \alpha Z_3 \rangle$ ($\alpha = \pm 1$);
6. $L_6 = \langle Z_3 \rangle$.

3. Reduction and exact solutions of Eq. (8)

We start with the reduction of Eq. (8) to ordinary differential equations. To this end we shall carry out a classification of one-dimensional subalgebras of the algebra $A(n,k)$ up to a contingency with respect to the group of inner automorphisms $\text{Ad}(\exp(\alpha X))$, where $X \in A(n,k)$. Using the method of the classification of subalgebras of finite-dimensional Lie algebras stated in [10–12] we receive the following theorem.

Equation (15) for $\alpha = 0$ has a solution

$$u^{1-k} = \frac{\lambda (k-1)^2}{4k} (z + C), \quad C = \text{const}$$

and the corresponding solution of Eq. (8)

$$u^{1-k} = \frac{\lambda (k-1)^2}{4k} (y_2 + Cy_1).$$
If \( z = \frac{k}{k+1} \) then Eq. (15) has the form
\[
- \frac{8l}{k+1} z\omega - \frac{4l}{k+1} \omega + \lambda \omega^k = 0
\]
the particular solution of which is the function
\[
\omega^{1-k} = \frac{\lambda (k-1)^2}{4l} (z^2 + C)^2, \quad C = \text{const.}
\]

Therefore Eq. (8) has the following solution
\[
u^{1-k} = \frac{\lambda (k-1)^2}{4l} \left( \frac{1}{y_2^2} + Cy_1 \right)^2. \tag{18}\]

(2) Subalgebra \( K_2 = \langle X_2 \rangle \). The ansatz
\[
u = \frac{1}{y_2^2} \omega(z), \quad z = y_1
\]
reduces (8) to the equation
\[- \frac{4}{k+1} z\omega - \frac{4l}{(k+1)^2} \omega + \lambda \omega^k = 0
\]
the solution of which is a function
\[
\omega^{1-k} = \frac{\lambda (k-1)^2}{4l} (1 + Cy_1^2), \quad C = \text{const.}
\]
From here it follows that the solution of Eq. (8) is a function
\[
u^{1-k} = \frac{\lambda (k-1)^2}{4l} y_2 (1 + Cy_1^2). \tag{19}\]

(3) Subalgebra \( K_3 = \langle X_1 + eX_3 \rangle \) \((e = \pm 1)\). The ansatz
\[
u = \frac{1}{y_1^2} \omega(z), \quad z = \frac{y_2}{y_1} - e \ln y_1
\]
corresponding to the given subalgebra reduces (8) to the equation
\[-4e\omega + \frac{4l}{k+1} \omega + \lambda \omega^k = 0.
\]

(4) Subalgebra \( K_4 = \langle M + \delta X_2 \rangle \) \((\delta = 0, \pm 1)\). The ansatz
\[
u = (y_1 y_2)^{\frac{1}{r}} \omega(z), \quad z = \frac{\delta}{l} y_1 + \ln y_2
\]
reduces (8) to the equation
\[-4\delta \omega + \frac{4l}{k+1} \omega + \lambda \omega^k = 0.
\]

(5) Subalgebra \( K_5 = \langle M + eX_3 \rangle \) \((e = \pm 1)\). The ansatz
\[
u = \frac{y_2}{y_1^2} \omega(z), \quad z = \frac{y_2}{y_1} + \frac{e}{l} y_1
\]
reduces (8) to the equation
\[-4e \omega + \lambda \omega^k = 0
\]
the particular solution of which is the function
\[
\omega^{1-k} = \frac{\lambda (k-1)^2}{8e(k+1)} (z + C)^2, \quad C = \text{const.}
\]
From here we obtain that the function

$$u^{1-k} = \frac{\lambda(k-1)^2}{8\epsilon(k+1)} y_1^{l-1} \left(y_2 + \epsilon y_1^{l-1} + Cy_1\right)^2$$

is a solution of Eq. (8).

Next we shall carry out the reduction of Eq. (8) for \( k = \frac{2}{n-2} \) and \( n > 2 \) with respect to one-dimensional subalgebras of the algebra \( A(n) \). The full list of such subalgebras are presented in Theorem 4. From this list it is necessary exclude the subalgebra \( L_6 = \langle Z_3 \rangle \) which has invariants \( u, y_1 \).

(1) Subalgebra \( L_1 = \langle Z_1 + \delta Z_2 \rangle \) (\( \delta = 0, \pm 1 \)). The ansatz

$$u = y_2^{\frac{1}{2\lambda}} \omega(z), \quad z = y_2 y_1^{l-1}$$

reduces (8) to the equation

$$-4\delta z^2 \ddot{\omega} + 2\delta(n-4)z \ddot{\omega} + \lambda \dot{\omega}^2 = 0.$$

(2) Subalgebra \( L_2 = \langle Z_2 \rangle \). The ansatz

$$u = y_2^{\frac{1}{2\lambda}} \omega(z), \quad z = y_1$$

reduces (8) to the equation

$$-2(n-2)z \ddot{\omega} + \lambda \dot{\omega}^2 = 0$$

the solution of which is the function

$$\omega^2 = -\frac{\lambda}{(n-2)^2} (\ln z + C), \quad C = \text{const}.$$

We get that the function

$$u^{2n} = -\frac{\lambda}{(n-2)^2} (y_2 \ln y_1 + Cy_2)$$

is a solution of Eq. (8).

(3) Subalgebra \( L_3 = \langle Z_1 + \varepsilon Z_3 \rangle \) (\( \varepsilon = \pm 1 \)). The ansatz

$$u = y_1^{\frac{1}{2\lambda}} \omega(z), \quad z = \frac{y_2}{y_1} - \varepsilon \ln y_1$$

reduces (8) to the equation

$$-4\varepsilon \ddot{\omega} + \lambda \dot{\omega}^2 = 0$$

the particular solution of which is the function

$$\omega^2 = \frac{\lambda}{4\varepsilon(n-1)(n-2)} (z + C)^2, \quad C = \text{const}.$$

Therefore the function

$$u^{2n} = \frac{\lambda}{4\varepsilon(n-1)(n-2)} y_1^{l-1} (y_2 - \varepsilon y_1 \ln y_1 + Cy_1)^2$$

is a solution of Eq. (8).

(4) Subalgebra \( L_4 = \langle S + \alpha Z_2 \rangle \) (\( \alpha \in \mathbb{R} \)). The ansatz

$$u = (y_1 \ln^{\frac{1}{2}+1} y_1)^{\frac{2\lambda}{\varepsilon}} \omega(z), \quad z = \frac{y_1 \ln^2 y_1}{y_2}$$

reduces (8) to the equation

$$-4\alpha z^3 \ddot{\omega} + 2(\alpha(n-4)x + n-2)z^2 \ddot{\omega} + \lambda \dot{\omega}^2 = 0.$$
If \( \rho = 0 \) then the function
\[
\omega q^0 = -\frac{\dot{\lambda}}{(n-2)^2}(z^{-1} + C), \quad C = \text{const}
\]
is a solution of Eq. (23). The corresponding solution of Eq. (8) has the form
\[
\omega q^0 = -\frac{\dot{\lambda}}{(n-2)^2}(y_2 + Cy_1) \ln y_1.
\]
(24)
In the case \( \rho = \frac{\dot{\lambda}}{2} \), \( n \neq 4 \) Eq. (23) has the solution
\[
\omega q^0 = -\frac{\dot{\lambda}}{(n-2)^2}z^{-1} \quad \text{which correspond to the solution} \quad \omega q^0 = -\frac{\dot{\lambda}}{(n-2)^2}y_2 \ln y_1 \quad \text{of Eq. (8)}.
\]
(25)

4. Generation of solutions

(a) Solutions of Eq. (6). Let us consider the task of multiplication of solutions (16)–(25) for Eq. (6). The one-parameter groups \( G_i \) generated by the operators \( X_i \) from the Theorem 1 and the one-parameter group \( G_0 \) which is generated by the operator \( M \) are the following
\[
G_0 : \{ y_1[1 - \partial y_1^{-1}], y_2[1 - \partial y_2^{1/2}], u[1 - \partial y_1^{1/2}] \},
\]
\[
G_1 : \left( e^{i} y_1, e^{i} y_2, y_1^{1/2}, u \right),
\]
\[
G_2 : \left( y_1, e^{i} y_2, y_1^{1/2}, u \right),
\]
\[
G_3 : \{ y_1, y_2, \partial y_1, u \}.
\]
The one-parameter groups \( G_i \) generated by the operators \( Z_i \) from the Theorem 2 and the one-parameter group \( G_0 \) which is generated by the operator \( S \) are the following
\[
G_0 : \left( y_1^{1/2}, y_2 y_1^{1/2}, e^{-\partial y_1^{1/2} y_1^{1/2} 1/2}, u \right),
\]
\[
G_1 : \left( y_1, e^{i} y_2, \partial y_1, u \right),
\]
\[
G_2 : \left( y_1, e^{i} y_2, y_1^{1/2}, u \right),
\]
\[
G_3 : \{ y_1, y_2, \partial y_1, u \}.
\]
For example, let us consider the one-parameter family of solutions of Eq. (6)
\[
u^{1-k} = \frac{i(k-1)^2}{4l}(y_2 + C_1 y_1).
\]
(28)
Any solution of the family (28) is invariant under the group \( G_1 \) and the transformations \( G_2, G_3 \) transfer the solutions of the family (28) again to the solutions of the same family. The group \( G_0 \) transfer the solutions of the family
\[
u^{1-k} = \frac{i(k-1)^2}{4l}(y_2 + C_1 y_1)(1 + C_2 y_1), \quad C_2 = \partial l.
\]
(29)
The obtained family (29) includes the family of solutions (28) and it is invariant under the transformations (26). Thus we get the family of solutions (29) of Eq. (6).

Analogously the family of the solutions (16) multiplies to the family
\[
u^{1-k} = \frac{i(k-1)^2}{4l} \left\{ (y_2 + C_1 y_1)(1 + C_2 y_1)^{1/2} + C_3 y_1^{1/2} y_1^{1/2 (k-1/2)} \right\}^2,
\]
(30)
and the family of solutions (18) – to the family
\[
u^{1-k} = \frac{i(k-1)^2}{4l} \left\{ (y_2 + C_1 y_1)(1 + C_2 y_1)^{1/2} + C_3 y_1^{1/2} (1 + C_2 y_1)^{1/2 (k-1/2)} \right\}^2,
\]
(31)
where \( C_1, C_2, C_3 \) are arbitrary constants.
The family of the solutions (19) multiplies to the family (29) and the family of solutions (20) – to the family
\[
u^{1-k} = \frac{i(k-1)^2}{8l(k+1)} C_1 y_1^{1/2} \left( y_2 + \frac{1}{C_1} y_1^{1/2} + C_2 y_1 \right)^2,
\]
(32)
where \( C_1, \ C_2 \) are arbitrary constants.
The families of the solutions (21), (24), (25) of Eq. (6) for \( k = \frac{n}{n-2} \) multiply to the family
\[
\nu^n = -\frac{\lambda}{(n-2)^2} (y_2 + C_1 y_1) (\ln y_1 + C_2),
\]
and the family of solutions (22) – to the family
\[
\nu^n = \frac{\lambda C_1}{4(n-1)(n-2)y_1} \left( y_2 - \frac{1}{C_1} y_1 \ln y_1 + C_2 y_1 \right)^2,
\]
where \( C_1, C_2 \) are arbitrary constants.

(b) Solutions of Eq. (1). The solutions of Eq. (1) we get from the formulae (29)–(34) setting \( n = 4, y_1 = x_3 - x_4, y_2 = x_1^2 + x_2^2 - x_3^2 - x_4^2 \) or \( n = 3, y_1 = x_3 - x_4, y_2 = x_2^2 - x_1^2 - x_3^2 \). Let us consider the first case in detail (explicitly). Taking into account \( n = 4 \) and \( l = k - 2 \) we obtain the following solutions of Eq. (1) in the case \( k \neq 2 \):

\[
\begin{align*}
\nu^{1-k} &= \frac{\lambda (k-1)^2}{4(k-2)} \left( y_2 + C_1 y_1 \right)(1 + C_2 y_1^{k-2}), \\
\nu^{1-k} &= \frac{\lambda (k-1)^2}{4(k-2)} \left( \left( y_2 + C_1 y_1 \right)(1 + C_2 y_1^{k-2}) \right)^{3\frac{k-1}{k-2}} + C_3 y_1^{3\frac{k-1}{k-2}} \left( 1 + C_2 y_1^{k-2} \right)^{\frac{k-1}{k-2}} y_1 \right)^2, \\
\nu^{1-k} &= \frac{\lambda (k-1)^2}{8(k+1)(k-2)} C_1 y_1^{k-3} \left( y_2 + \frac{1}{C_1} y_1^{2-k} + C_2 y_1 \right)^2.
\end{align*}
\]

If \( k = 2 \) then we get such solutions of Eq. (1)
\[
\begin{align*}
\nu^{-1} &= -\frac{\lambda}{4} (y_2 + C_1 y_1) (\ln y_1 + C_2), \\
\nu^{-1} &= \frac{\lambda C_1}{24y_1} \left( y_2 - \frac{1}{C_1} y_1 \ln y_1 + C_2 y_1 \right)^2.
\end{align*}
\]

5. Generalization of solutions

Evidently the family of solutions (35) is a particular case of the family (36) if in it we set \( C_3 = 0 \). The solutions (36)–(40) were obtained using the ansatz (4) which can be generalized in this way. Let \( h(x_1, x_2, x_3, x_4) \) is a function which satisfies the system of the equations
\[
\begin{align*}
\Delta h &= 0, \quad (\nabla h)^2 = 0, \quad \nabla h \cdot \nabla y_1 = 0, \quad \nabla h \cdot \nabla y_2 = 0,
\end{align*}
\]
where \( y_1 = x_1 - x_4, y_2 = x_1^2 + x_2^2 - x_3^2 - x_4^2 = 0 \). The general solution of the system (41) has the form \( h = \varphi(y_3) \) where \( y_3 = (x_1 - x_4)(x_2 - x_3)^{-1} \) and \( \varphi(h) \) is an arbitrary smooth function of \( y_3 \). Then the ansatz
\[
\begin{align*}
\nu = u(y_1, y_2) = y_1 = x_1 - x_4, \\
y_2 = x_1^2 + x_2^2 - x_3^2 - x_4^2, \quad y_3 = (x_1 - x_4)(x_2 - x_3)^{-1}
\end{align*}
\]
reduces Eq. (1) to the same equation that the ansatz \( u = u(y_1, y_2) \). Thus on the basis of the work we get the wider class of the solutions of Eq. (1) if in the solutions (36)–(40) the constants \( C_1, C_2, C_3 \) arbitrary functions of \( y_3 = (x_1 - x_4)(x_2 - x_3)^{-1} \).

The solutions (36)–(40) of Eq. (1) \( C_1, C_2, C_3 \) are arbitrary functions of \( y_3 \) can be multiply with the help of the invariance group of Eq. (1). Let
\[
a = (a_1, a_2, a_3, a_4), \quad b = (b_1, b_2, b_3, b_4), \quad z = (z_1, z_2, z_3, z_4),
\]
\[
z_i = x_i + x_i, \quad (a, b) = a_1 b_1 + a_2 b_2 - a_3 b_3 - a_4 b_4, \quad h = (a, z)(b, z)^{-1},
\]
where \( (a, a) = (a, b) = (b, b) = 0, \varphi(h) \) are arbitrary smooth functions of \( h, a, b, x_i \in \mathbb{R} \) \( i = 1, 2, 3, 4 \).
The family of the solutions \((36)-(40)\) multiplies with the help of the invariance group of Eq. (1) to following families of solutions:

\[
u^{1-k} = \frac{i(k-1)^2}{4(k-2)} \left[ \left( (z, z) + \varphi_1(h)(a, z) \right) \left( 1 + \varphi_2(h)(a, z)^{-k} \right) \right]^2 + \varphi_2(h) (a, z)^{\frac{1}{k+1}} (z, z) + \varphi_1(h)(a, z)^{\frac{1}{k+1}} \right]^2 \quad (k \neq 2), (42)\]

\[
u^{1-k} = \frac{i(k-1)^2}{4(k-2)} \left[ \left( (z, z) + \varphi_1(h)(a, z) \right) \left( 1 + \varphi_2(h)(a, z)^{-k} \right) \right]^2 + \varphi_2(h) (a, z)^{\frac{1}{k+1}} (z, z) + \varphi_1(h)(a, z)^{\frac{1}{k+1}} \right]^2 \quad (k \neq 2), (43)\]

\[
u^{-1} = \frac{\varphi_1(h)}{4} \left( (z, z) + \varphi_1(h)(a, z) \right) \ln(a, z) + \varphi_2(h) \quad (k = 2), (45)\]

\[
u^{-1} = \frac{\varphi_1(h)}{24} \left( (z, z) + \frac{1}{\varphi_1(h)} (a, z) \ln(a, z) + \varphi_2(h)(a, z) \right)^2 \quad (k = 2), (46)\]

where \(\varphi_1, \varphi_2, \varphi_3\) are arbitrary functions of \(h\).

Setting in the formulae \((29)-(34)\) \(n = 3, y_1 = x_2 - x_1, y_2 = x_2^2 - x_3^2 - x_4^2\) we obtain such families of the solutions of Eq. (1)

\[
u^{1-k} = \frac{i(k-1)^2}{2(k-3)} \left( y_2 + C_1 y_1 \right) \left( 1 + C_2 y_1^{\frac{k+1}{k-1}} \right)^2 \quad (k \neq 3), (42)\]

\[
u^{1-k} = \frac{i(k-1)^2}{2(k-3)} \left( y_2 + C_1 y_1 \right) \left( 1 + C_2 y_1^{\frac{k+1}{k-1}} \right)^2 \quad (k \neq 3), (43)\]

\[
u^{-1} = \frac{i(k-1)^2}{4(k-3)(k+1)} \left( C_1 y_1^{\frac{k+1}{k-1}} (y_2 + \frac{1}{C_1} y_1^{\frac{k+1}{k-1}} \right)^2 \quad (k \neq 3), (45)\]

\[
u^{2} = -\frac{C_1}{C_1} \left( y_2 - \frac{y_1}{C_1} \right) \ln y_1 + C_2 \quad (k = 3), (46)\]

\[
u^{2} = -\frac{C_1}{8} \left( y_2 - \frac{y_1}{C_1} \right) \ln y_1 + C_2 y_1 \quad (k = 3). (46)\]

6. Exact solutions of d'Alembert equation

In the case \(k = 3\) Eq. (1) is invariant under conformal algebra \(AC(2, 2)\) the base of which is generated by operators \((2)\) (for \(k = 3\)) and \((3)\). The three families of solutions are obtained from the formulae \((42)-(44)\), if we put \(k = 3\). The construction other classes of exact solutions we begin simple ansatz \(\omega = \omega(z), z = x_1\) which reduces Eq. (1) to the ordinary differential equation

\[
\ddot{\omega} + \omega \dot{\omega}^3 = 0. \tag{47}
\]

If \(\dot{\omega} < 0\) then the substitution \(\omega = \sqrt{-\frac{1}{2}} \nu\) reduced Eq. (47) to the form

\[
\ddot{\nu} = 2 \nu^3 \tag{48}
\]

and in the case \(\dot{\omega} > 0\) the substitution \(\omega = \sqrt{\frac{1}{2}} \nu\) reduced Eq. (47)

\[
\ddot{\nu} = -2 \nu^3. \tag{49}
\]

Eq. (48) is equivalent to the equation

\[
\dot{\nu}^2 = \nu^4 + C_n, \tag{50}
\]

where \(C_n\) is integration constant. If \(C_n = -\frac{1}{4}\) then Eq. (50) has a solution

\[
\nu = ds \left( z, \frac{1}{\sqrt{2}} \right), \tag{51}
\]

where \(ds(y, k)\) is the Jacobi elliptic function satisfying the equation

\[
\left( \frac{dn}{dy} \right)^2 = k^2 (k^2 - 1) + (2k^2 - 1) \eta^4 + \eta^4. \tag{50}
\]

The Eq. (50) was investigated in [9], where the following properties of solutions are determined:
(a) If $u^{(n)}$ is a solution of Eq. (50) then the function

$$u^{(n+1)} = \frac{\dot{u}^{(n)}}{u^{(n)}}, \quad \text{where} \quad \dot{u}^{(n)} = \frac{d u^{(n)}}{dz},$$

is also a solution of Eq. (50), moreover

$$(\dot{u}^{(n+1)})^2 = (\dot{u}^{(n+1)})^4 + C_{n+1}, \quad C_{n+1} = 4C_n.$$  

(b) If $u^{(n)}$ is a solution of Eq. (50) for $C_n > 0$ then the function

$$\ddot{u}^{(n)} = \frac{\sqrt{C_n}}{\dot{u}^{(n)}}$$

is also a solution of Eq. (50).

(c) Let $u^{(n)}$ is a solution of Eq. (50) for $C_n = -B_n < 0$ then the function

$$\ddot{u}^{(n)} = \frac{\sqrt{-B_n}}{\dot{u}^{(n)}}$$

is a solution of Eq. (49) and let additionally

$$(\dot{u}^{(n)})^2 = -(\dot{u}^{(n)})^4 + B_n^2.$$  

Using properties (a), (b) we obtain infinite families of solutions of Eq. (1) for $k = 3$ and $\lambda < 0$

$$u^{(n)} = \sqrt{-\frac{2}{\lambda}}v^{(n)}, \quad n = 0, 1, 2, \ldots,$$

$$\tilde{u}^{(2n+1)} = \sqrt{-\frac{2}{\lambda}}v^{(2n+1)}, \quad r = 0, 1, 2, \ldots,$$

where the function $\tilde{u}^{(2n+1)}$ is defined by (53) and the function $u^{(n)}$ satisfies the recurrence relations

$$u^{(n)} = \frac{v^{(n-1)}}{v^{(n-1)}}, \quad \nu^{(0)} = ds\left(z, \frac{1}{\sqrt{2}}\right).$$

For $n = 0, 1, 2, 3, \ldots$ we obtain from (55) and (56) solutions defined by the formulae

$$u^{(0)} = \sqrt{-\frac{2}{\lambda}}ds\left(z, \frac{1}{\sqrt{2}}\right), \quad u^{(1)} = -\sqrt{-\frac{2}{\lambda}}\frac{cs\left(z, \frac{1}{\sqrt{2}}\right)}{sn\left(z, \frac{1}{\sqrt{2}}\right)},$$

$$u^{(2)} = \sqrt{-\frac{2}{\lambda}}\left[cd\left(z, \frac{1}{\sqrt{2}}\right) - dc\left(z, \frac{1}{\sqrt{2}}\right) - cn\left(z, \frac{1}{\sqrt{2}}\right) ds\left(z, \frac{1}{\sqrt{2}}\right)\right],$$

$$u^{(3)} = \sqrt{-\frac{2}{\lambda}}\frac{cs\left(z, \frac{1}{\sqrt{2}}\right)}{sn\left(z, \frac{1}{\sqrt{2}}\right)} ds\left(z, \frac{1}{\sqrt{2}}\right) dr\left(z, \frac{1}{\sqrt{2}}\right) - \frac{cn\left(z, \frac{1}{\sqrt{2}}\right) ds\left(z, \frac{1}{\sqrt{2}}\right) dr\left(z, \frac{1}{\sqrt{2}}\right)}{sn\left(z, \frac{1}{\sqrt{2}}\right)},$$

$$\tilde{u}_1 = \sqrt{-\frac{2}{\lambda}}\frac{ds\left(z, \frac{1}{\sqrt{2}}\right)}{cs\left(z, \frac{1}{\sqrt{2}}\right)},$$

$$\tilde{u}_3 = 4\sqrt{-\frac{2}{\lambda}}\frac{ds\left(z, \frac{1}{\sqrt{2}}\right)}{cs\left(z, \frac{1}{\sqrt{2}}\right)} ds\left(z, \frac{1}{\sqrt{2}}\right) dr\left(z, \frac{1}{\sqrt{2}}\right) - \frac{cn\left(z, \frac{1}{\sqrt{2}}\right) ds\left(z, \frac{1}{\sqrt{2}}\right) dr\left(z, \frac{1}{\sqrt{2}}\right)}{sn\left(z, \frac{1}{\sqrt{2}}\right)}.$$  

In the formulae (58), (59) $z = x_1$, i.e. we obtained the solutions of d'Alembert equation which depend on the only variable $x_1$. Having operated the one-parameter group $exp(\theta P_1)$ we get the family of the solutions (58), (59) where $z = x_1 + \theta, \theta$ is a constant. Using the method of constructing of solutions proposed in [4] these solutions can be generalized replaced $\theta$ by an arbitrary solution $h = h(x_2, x_3, x_4)$ of Smirnov–Sobolev system [13–15].
Thus the formulae (58), (59) where \( z = x_1 + h(x_2, x_3, x_4) \) and \( h(x_2, x_3, x_4) \) is an arbitrary solution of the system (60) determine wide classes of exact solutions of d’Alembert equation in the case \( k = 3, \lambda < 0 \).

The property (c) gives a possibility to construct infinity sets of exact solutions of Eq. (49). The exact solutions of Eq. (49) have the form (52) where \( u^{(n)} \) are solutions (57) for even \( n \). The corresponding list of the exact solutions of the Eq. (1) for the case \( \lambda > 0 \) are determined by the following formulae:

\[
\begin{align*}
\hat{u}_0 &= \frac{1}{2} \sqrt{2} \text{sd} \left( z, \frac{1}{\sqrt{2}} \right), \\
\hat{u}_2 &= 2 \sqrt{2} \text{cd} \left( z, \frac{1}{\sqrt{2}} \right) - \text{dn} \left( z, \frac{1}{\sqrt{2}} \right) \text{sn} \left( z, \frac{1}{\sqrt{2}} \right),
\end{align*}
\]

Setting in the formulae (61) \( z = x_1 + h(x_2, x_3, x_4) \) where \( h(x_2, x_3, x_4) \) is an arbitrary solution of the system (60) we obtain wide classes of exact solutions of d’Alembert equation in the case \( k = 3, \lambda > 0 \).

Now for constructing of solutions of the Eq. (1) we use the symmetry operator \( K_x \). The group \( G_x \) that correspond to this operator has the form

\[
x_1 \rightarrow x_1 + \theta(x_1^2 + x_2^2 - x_3^2 - x_4^2), \\
x_i \rightarrow x_i + \theta(x_i^2 + x_j^2 - x_k^2 - x_l^2) \quad (i = 2, 3, 4), \\
u \rightarrow u + 1 - 2 \theta(x_1^2 + x_2^2 - x_3^2 - x_4^2) \quad (i = 2, 3, 4).
\]

Let \( \lambda < 0 \) and \( u^{(0)} = \sqrt{-2} \text{ds} \left( z, \frac{1}{\sqrt{2}} \right) \) is a solution of Eq. (1). Having operated the transformation \( G_x \) we get a solution

\[
u^{(0)} = [1 - 2 \theta(x_1^2 + x_2^2 - x_3^2 - x_4^2)]^{-1} \text{ds} \left( z, \frac{1}{\sqrt{2}} \right),
\]

where

\[
z = \frac{x_1 + \theta(x_1^2 + x_2^2 - x_3^2 - x_4^2)}{1 - 2 \theta(x_1^2 + x_2^2 - x_3^2 - x_4^2)}.
\]

Let \( y_1 = x_1, y_2 = x_2 - x_3 - x_4 \) and \( h = h(x_1, x_2, x_3, x_4) \). If the function \( h \) satisfies the system

\[
\Box h = 0, \quad \nabla h \cdot \nabla y_1 = 0, \quad \nabla h \cdot \nabla y_2 = 0,
\]

then it has the form \( h = \varphi(y) \) where \( \varphi \) is an arbitrary smooth function of a variable

\[
y = \frac{x_3 x_4 \pm \sqrt{x_3^2 x_4^2 + x_2^2 x_4^2 - x_2^2}}{x_2^2 - x_4^2}.
\]

Having replaced in (62), (63) the constant \( \theta \) by an arbitrary function \( \varphi \) of the variable \( (64) \) we get a wide class of solutions of d’Alembert equation in the case \( \lambda < 0 \). In analogous way can be generalized other solutions.

Let us return to a consideration the subalgebra \( K_5 = \langle M + \varepsilon X_3 \rangle \) \((\varepsilon = \pm 1)\) from Theorem 3. For \( k = 3 \) the ansatz corresponding to \( K_6 \) has the form

\[
u = y_1^{-1} \varphi(z), \quad z = \frac{y_2}{y_1} + \varepsilon y_1^{-1}
\]

and reduce the Eq. (6) to the equation

\[
-4\varepsilon \varphi_0 + \lambda \varphi^2 = 0.
\]

The constructing solutions of Eq. (6) invariant under \( K_5 \) comes to the constructing solutions of Eq. (65). Using the formulae (55), (56) we get the following infinity sets of exact solutions for Eq. (1) in the case \( k = 3, \varepsilon = -1, \lambda < 0 \):

\[
u^{(n)} = \sqrt{-\frac{8}{\lambda} y_1^{-1} v^{(n)}(z)}, \quad n = 0, 1, 2, \ldots
\]

\[
u^{(2r+1)} = 2^r \sqrt{-\frac{8}{\lambda} y_1^{-1} \tilde{v}^{(2r+1)}(z)}, \quad r = 0, 1, 2, \ldots
\]

where the function \( \tilde{v}^{(2r+1)} \) is defined by the formula (51) and the function \( v^{(n)} \) satisfies the recurrence relation (57). So for \( n = 0 \)
\[ u^{(0)} = \sqrt{-\frac{8}{\lambda}} y_1^{-1} \, ds \left( z, \frac{1}{\sqrt{2}} \right). \]

and using one-parametric group \( G_0 = \exp(\theta M) \) we obtain the following solution
\[ u^{(0)} = \sqrt{-\frac{8}{\lambda}} y_1^{-1} \left( \frac{y_2 [1 - \theta y_1]^{-1} - 1}{y_1 [1 - \theta y_1]^{-1}} \right). \]

After substitution instead constant \( \theta \) an arbitrary function \( \varphi(y_3), y_3 = (x_1 - x_4)(x_2 - x_3)^{-1} \), we find the next family of solution
\[ u^{(0)} = \sqrt{-\frac{8}{\lambda}} y_1^{-1} ds \left( y, \frac{1}{\sqrt{2}} \right), \]

where
\[ y = \frac{y_2 [1 + \varphi(y_3)] y_1^{-1} - 1}{y_1 [1 + \varphi(y_3)] y_1^{-1}}. \] (68)

Similarly we can generalize all other solutions of series (66), (67).

Once more infinite class of solution we have obtained in the case \( \varepsilon = 1, \lambda > 0 \) by using property (c)
\[ \tilde{u}^{(n)} = \sqrt{\frac{8}{\lambda}} y_1^{-1} \tilde{v}^{(n)}(y), \quad n = 0, 2, 4, \ldots, \]

where functions \( \tilde{v}^{(n)} \) are defined by (53) and (68).

References