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A NEW METHOD FOR THE CONSTRUCTION OF SOLUTIONS OF NONLINEAR WAVE EQUATIONS

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We propose a simple new method for the construction of solutions of multidimensional nonlinear wave equations.

1. Introduction

One efficient method for the construction of solutions of nonlinear equations of mathematical physics is the method of symmetry reduction to equations with a smaller number of variables, in particular, to ordinary differential equations [1–3]. This method is based on the investigation of the subgroup structure of the invariance group of the given differential equation. The solutions thus obtained are invariant with respect to a subgroup of the invariance group of the equation. It should be noted that the invariance imposes very strict restrictions on solutions. Therefore, in many cases, the symmetry reduction does not allow one to obtain sufficiently broad classes of solutions.

In [3–9], the idea of conditional invariance of differential equations was proposed. The conditional invariance is understood as the symmetry of a certain subset of solutions. Many important nonlinear equations of mathematical physics have subsets of solutions whose symmetry substantially differs from that of the entire set of solutions. Such subsets are usually specified by additional conditions, which are certain partial differential equations. The explicit description of these additional conditions is a complicated problem and, unfortunately, there are no efficient methods for its solution.

In the present paper, we suggest a constructive method for finding certain classes of exact solutions of multidimensional nonlinear wave equations. The idea of the method is as follows: Consider a partial differential equation

$$F(x, \underset{1}{u}, \underset{2}{u}, \ldots, \underset{m}{u}) = 0, \tag{1}$$

where $u = u(x)$, $x = (x_0, x_1, \ldots, x_n) \in R_{1,n}$, and $\underset{m}{u}$ is the collection of all possible mth-order derivatives. We assume that Eq. (1) has a nontrivial symmetry algebra. To construct solutions of Eq. (1), we use a symmetry (or conditional-symmetry) ansatz [3]. We assume that it has the form

$$u = f(x)\,\varphi(\omega_1, \ldots, \omega_k) + g(x), \tag{2}$$

where $\omega_1 = \omega_1(x_0, x_1, \ldots, x_k), \ldots, \omega_k = \omega_k(x_0, x_1, \ldots, x_k)$ are new independent variables. Ansatz (2) selects a certain subset S of the entire set of solutions of Eq. (1). Let us construct, if possible, a new ansatz

$$u = f(x)\,\varphi(\omega_1, \ldots, \omega_k, \omega_{k+1}, \ldots, \omega_L) + g(x), \tag{3}$$

which is a generalization of ansatz (2). Here, $\omega_{k+1}, \ldots, \omega_L$ are new variables to be determined. We determine the variables $\omega_{k+1}, \ldots, \omega_L$ from the condition that the reduced equation corresponding to ansatz (3) coincides with the reduced equation corresponding to ansatz (2). Ansatz (3) selects a subset S_1 of the set of solutions of Eq. (1),

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which is an extension of the subset S. If the solutions belonging to the subset S are known, then one can construct the solutions from the subset S_1. These solutions are constructed as follows: Let $u = u(x, C_1, \ldots, C_t)$ be a multiparameter family of solutions of Eq. (1) that have the form (2); here, $C_1, \ldots, C_t$ are arbitrary constants. Then we obtain a more general family of solutions of Eq. (1) if the constants C_i in the solution $u = u(x, C_1, \ldots, C_t)$ are regarded as arbitrary smooth functions of $\omega_{k+1}, \ldots, \omega_L$.

We note that the idea of this method was formulated in [7] and developed in [8, 9]. In order to apply this method to finding exact solutions of nonlinear equations of mathematical physics, one should use ansatzes (2), an algorithm for the construction of which is not indicated in [7]. In the present paper, which is a logical continuation of [7–9], the method indicated is realized for the nonlinear d'Alembert and eikonal equations as well as for Schrödinger-type equations. By using the subgroup structure of the invariance groups of the equations considered [10–12], we obtain efficient ansatzes that allow one to construct broad new classes of exact solutions containing arbitrary functions.

2. Nonlinear d'Alembert Equation

We consider a nonlinear Poincaré-invariant d'Alembert equation

$$\Box u + F(u) = 0, \tag{4}$$

where

$$\Box u = \frac{\partial^2 u}{\partial x_0^2} - \frac{\partial^2 u}{\partial x_1^2} - \ldots - \frac{\partial^2 u}{\partial x_n^2}$$

and $F(u)$ is an arbitrary smooth function. The construction of exact solutions of Eq. (4) with various conditions imposed on the function $F(u)$ is considered in [3, 11–13]. The majority of these solutions are invariant with respect to the invariance group of Eq. (4), i.e., they are Lie solutions. One of the methods for the construction of solutions is the method of symmetry reduction of Eq. (4) to ordinary differential equations. The idea of this method for Eq. (4) is as follows.

Equation (4) is invariant with respect to the Poincaré algebra $AP(1, n)$ with basis elements

$$J_{0a} = x_0\partial_a + x_a\partial_0, \qquad J_{ab} = x_b\partial_a - x_a\partial_b,$$

$$P_0 = \partial_0, \quad P_a = \partial_a, \qquad a, b = 1, 2, \ldots, n.$$

Let L be an arbitrary subalgebra of rank n of the algebra $AP(1, n)$. The subalgebra L has two basic invariants u and $\omega = \omega(x_0, x_1, \ldots, x_n)$. The ansatz $u = \varphi(\omega)$ that corresponds to the subalgebra L reduces Eq. (4) to the ordinary differential equation

$$\ddot{\varphi}(\nabla\omega)^2 + \dot{\varphi}\Box\omega + F(\varphi) = 0, \tag{5}$$

$$(\nabla\omega)^2 = \left(\frac{\partial\omega}{\partial x_0}\right)^2 - \left(\frac{\partial\omega}{\partial x_1}\right)^2 - \ldots - \left(\frac{\partial\omega}{\partial x_n}\right)^2.$$

Such a reduction (as well as the corresponding ansatz) is called a symmetry reduction. There are eight types of nonequivalent subalgebras of rank n in the algebra $AP(1, n)$ [11].

In [14], the following procedure for the reduction of Eq. (4) to ordinary differential equations, which is a generalization of the method of symmetry reduction, was suggested: By using an ansatz $u = \varphi(\omega)$, where $\omega = \omega(x)$ is a new variable, Eq. (4) is reduced to an ordinary differential equation if $\omega(x)$ satisfies the equations

$$\begin{aligned} \Box\omega &= F_1(\omega), \\ (\nabla\omega)^2 &= F_2(\omega). \end{aligned} \tag{6}$$

Here, F_1 and F_2 are arbitrary smooth functions that depend only on ω.

Thus, having constructed all solutions of system (6), one obtains the collection of variables ω for which the ansatz $u = \varphi(\omega)$ reduces Eq. (4) to an ordinary differential equation for ω. System (6) is studied in [4, 15].

However, it should be noted that the ansatzes obtained from system (6) do not exhaust all ansatzes that reduce Eq. (4) to ordinary differential equations. In this connection, we now consider the problem of finding generalized ansatzes (3) by using given symmetry ansatzes (2) of Eq. (4).

2.1. Consider a symmetry ansatz $u = \varphi(\omega_1)$ for Eq. (4), where $\omega_1 = (x_0^2 - x_1^2 - \ldots - x_k^2)^{1/2}$, $k \geq 2$. This ansatz reduces Eq. (4) to the equation

$$\varphi_{11} + \frac{k}{\omega_1}\varphi_1 + F(\varphi) = 0, \tag{7}$$

where $\varphi_{11} = \dfrac{d^2\varphi}{d\omega_1^2}$, $\varphi_1 = \dfrac{d\varphi}{d\omega_1}$. This ansatz should be regarded as a particular case of the more general ansatz $u = \varphi(\omega_1, \omega_2)$, where ω_2 is an unknown variable. The ansatz $u = \varphi(\omega_1, \omega_2)$ reduces the equation

$$\varphi_{11} + \frac{k}{\omega_1}\varphi_1 + 2\varphi_{12}(\nabla\omega_1\cdot\nabla\omega_2) + \varphi_2\Box\omega_2 + \varphi_{22}(\nabla\omega_2)^2 + F(\varphi) = 0, \tag{8}$$

$$\nabla\omega_1\cdot\nabla\omega_2 = \frac{\partial\omega_1}{\partial x_0}\frac{\partial\omega_2}{\partial x_0} - \frac{\partial\omega_1}{\partial x_1}\frac{\partial\omega_2}{\partial x_1} - \ldots - \frac{\partial\omega_1}{\partial x_n}\frac{\partial\omega_2}{\partial x_n}.$$

On Eq. (8), we impose the condition that it should coincide with the reduced equation (7). Under this assumption, Eq. (8) splits into the following two equations:

$$\varphi_{11} + \frac{k}{\omega_1}\varphi_1 + F(\varphi) = 0, \tag{9}$$

$$2\varphi_{12}(\nabla\omega_1\cdot\nabla\omega_2) + \varphi_{22}(\nabla\omega_2)^2 + \varphi_2\Box\omega_2 = 0. \tag{10}$$

Equation (10) is satisfied for an arbitrary function φ if the variable ω_2 is such that

$$\Box\omega_2 = 0, \quad (\nabla\omega_2)^2 = 0, \tag{11}$$

$$\nabla\omega_1\cdot\nabla\omega_2 = 0. \tag{12}$$

Thus, if the variable ω_2 is chosen so that conditions (11) and (12) are satisfied, the multidimensional equation (4) reduces to the ordinary differential equation (7), and its solutions give solutions of Eq. (4). The problem of reduction thus turns into the problem of construction of general or partial solutions of system (11), (12).

The overdetermined system (11) is studied in detail in [16, 17], where a broad class of solutions of system (11) is obtained. These solutions are constructed as follows: We consider the linear algebraic equation for the variables $x_0, x_1, \ldots, x_n$ with coefficients depending on the unknown ω_2:

$$a_0(\omega_2)x_0 - a_1(\omega_2)x_1 - \ldots - a_n(\omega_2)x_n - b(\omega_2) = 0. \tag{13}$$

Let the coefficients of this equation be analytic functions of ω_2 satisfying the condition

$$[a_0(\omega_2)]^2 - [a_1(\omega_2)]^2 - \ldots - [a_n(\omega_2)]^2 = 0.$$

Assume that Eq. (13) is solvable with respect to ω_2 and the result of its solution is a certain function

$$\omega_2(x_0, x_1, \ldots, x_n), \tag{14}$$

which may be real or complex. Then, function (14) is a solution of system (11). We now select the solutions of (14) that have the additional property $\nabla\omega_1 \nabla\omega_2 = 0$. It is obvious that

$$\frac{\partial\omega_2}{\partial x_0} = -\frac{a_0}{\delta'}, \quad \frac{\partial\omega_2}{\partial x_1} = \frac{a_1}{\delta'}, \quad \ldots, \quad \frac{\partial\omega_2}{\partial x_n} = \frac{a_n}{\delta'},$$

where $\delta(\omega_2) \equiv a_0(\omega_2)x_0 - a_1(\omega_2)x_1 - \ldots - a_n(\omega_2)x_n - b(\omega_2)$ and δ' is the derivative of δ with respect to ω_2. Since

$$\frac{\partial\omega_1}{\partial x_0} = \frac{x_0}{\omega_1}, \quad \frac{\partial\omega_1}{\partial x_1} = -\frac{x_1}{\omega_1}, \ldots, \quad \frac{\partial\omega_1}{\partial x_n} = -\frac{x_n}{\omega_1},$$

we have

$$\nabla\omega_1 \cdot \nabla\omega_2 = -\frac{1}{\omega_1 \delta'}(a_0x_0 - a_1x_1 - \ldots - a_nx_n).$$

In view of (13), the equality $\nabla\omega_1 \cdot \nabla\omega_2 = 0$ holds if and only if $b(\omega_2) = 0$. Thus, we have constructed a broad class of ansatzes that reduce the d'Alembert equation to ordinary differential equations. The arbitrariness of the choice of the function ω_2 may be used to satisfy certain additional conditions.

2.2. The symmetry ansatz $u = \varphi(\omega_1)$, $\omega_1 = (x_1^2 + \ldots + x_L^2)^{1/2}$, $L \geq 1$, $L < n - 1$, is generalized as follows: Let ω_2 be an arbitrary solution of the system of equations

$$\begin{gathered} \frac{\partial^2\omega}{\partial x_0^2} - \frac{\partial^2\omega}{\partial x_{L+1}^2} - \ldots - \frac{\partial^2\omega}{\partial x_n^2} = 0, \\ \left(\frac{\partial\omega}{\partial x_0}\right)^2 - \left(\frac{\partial\omega}{\partial x_{L+1}}\right)^2 - \ldots - \left(\frac{\partial\omega}{\partial x_n}\right)^2 = 0. \end{gathered} \tag{15}$$

The ansatz $u = \varphi(\omega_1, \omega_2)$ reduces Eq. (4) to the equation

$$-\frac{\partial^2\varphi}{\partial\omega_1^2}-\frac{L-1}{\omega_1}\frac{\partial\varphi}{\partial\omega_1}+F(\varphi) = 0.$$

If $L = n-1$, then the ansatz $u = \varphi(\omega_1, \omega_2)$, $\omega_2 = x_0 - x_n$, is a generalization of the symmetry ansatz $u = \varphi(\omega_1)$.

The ansatzes corresponding to subalgebras 2, 6, and 8 in Table 1 in [9] are particular cases of the ansatz constructed above. Similarly, one can obtain broad classes of ansatzes that reduce Eq. (4) to at least two-dimensional equations. Let us now present some of them.

2.3. The ansatz $u = \varphi(\omega_1,\dots,\omega_L,\omega_{L+1})$, where $\omega_1 = x_1, \dots, \omega_L = x_L$, ω_{L+1} is an arbitrary solution of system (15) and $L \le n-1$, is a generalization of the symmetry ansatz $u = \varphi(\omega_1, \dots, \omega_L)$ and reduces Eq. (4) to the equation

$$-\frac{\partial^2\varphi}{\partial\omega_1^2}-\frac{\partial^2\varphi}{\partial\omega_2^2}-\dots-\frac{\partial^2\varphi}{\partial\omega_L^2}+F(\varphi) = 0.$$

2.4. Let $\omega_1 = (x_1^2 - x_2^2 - \dots - x_L^2)^{1/2}, \omega_2 = x_{L+1}, \dots, \omega_s = x_{L+s-1}$, $L \ge 2$, $L+s-1 \le n$, ω_{s+1} be an arbitrary solution of the system

$$\begin{gathered}\Box\omega_{s+1} = 0, \quad (\nabla\omega_{s+1})^2 = 0,\\ \nabla\omega_i\cdot\nabla\omega_{s+1} = 0, \quad i = 1, 2, \dots, s.\end{gathered} \tag{16}$$

Then the ansatz $u = \varphi(\omega_1,\dots,\omega_s,\omega_{s+1})$ is a generalization of the symmetry ansatz $u = \varphi(\omega_1, \dots, \omega_s)$ that reduces Eq. (4) to the equation

$$\varphi_{11}-\frac{L}{\omega_1}\varphi_1-\varphi_{22}-\dots-\varphi_{ss}+F(\varphi) = 0.$$

3. Exact Solutions of the Nonlinear d'Alembert Equation

Let us construct some classes of exact solutions of the equation

$$\Box u + \lambda u^k = 0, \quad k \ne 1. \tag{17}$$

Consider the invariant solution [12] of Eq. (17):

$$u^{1-k} = \sigma(k, L)(x_1^2 + \dots + x_L^2), \tag{18}$$

$$\sigma(k, L) = \frac{\lambda(1-k)^2}{2(L-Lk+2k)}, \quad L = 1, 2, \dots, n.$$

Applying the group transformation to solution (18), we obtain the multiparameter family of solutions

$$u^{1-k} = \sigma(k, L)[(x_1 + C_1)^2 + \dots + (x_L + C_L)^2],$$

where $C_1, \ldots, C_L$ are arbitrary constants. Hence, according to Sec. 2.3, for $L \leq n-1$ we obtain the following family of solutions of Eq. (17):

$$u^{1-k} = \sigma(k, L)[(x_1 + h_1(\omega))^2 + \ldots + (x_L + h_L(\omega))^2], \qquad k \neq \frac{L}{L-2},$$

where ω is an arbitrary solution of system (15) and $h_1(\omega), \ldots, h_L(\omega)$ are arbitrary twice-differentiable functions of ω. If, in particular, $n = 3$ and $L = 1$, then Eq. (17) has the following family of solutions in the space $R_{1,3}$:

$$u^{1-k} = \frac{\lambda(L-k)^2}{2(1+k)}[x_1 + h_1(\omega)]^2, \qquad k \neq -1.$$

Consider the following solution [12] of Eq. (17):

$$u^{1-k} = \sigma(k, s)(x_0^2 - x_1^2 - \ldots - x_s^2), \qquad s = 2, \ldots, n, \tag{19}$$

$$\sigma(k, s) = -\frac{\lambda(1-k)^2}{2(s - ks + k + 1)}, \qquad k \neq \frac{s+1}{s-1}.$$

Solution (19) determines the multiparameter family of solutions

$$u^{1-k} = \sigma(k, s)[x_0^2 - x_1^2 - x_L^2 - (x_{L+1} + C_{L+1})^2 - \ldots - (x_s + C_s)^2],$$

where $C_{L+1}, \ldots, C_s$ are arbitrary constants. Hence, according to Sec. 2.4, for $L \geq 2$ we obtain the family of solutions

$$u^{1-k} = \sigma(k, s)[x_0^2 - x_1^2 - x_L^2 - (x_{L+1} + h_{L+1}(\omega))^2 - \ldots - (x_s + h_s(\omega))^2],$$

where ω is an arbitrary solution of system (16) and $h_{L+1}(\omega), \ldots, h_s(\omega)$ are arbitrary twice-differentiable functions of ω. If, in particular, $L = 2$ and $s = 3$, then Eq. (17) has the following family of solutions in the space $R_{1,3}$:

$$u^{1-k} = \frac{\lambda(L-k)^2}{4(k-2)}[x_0^2 - x_1^2 - x_2^2 - (x_3 + h_3(\omega))^2], \qquad k \neq 2.$$

The equation

$$\Box u + 6u^2 = 0 \tag{20}$$

has the solution $u = \mathcal{P}(x_3 + C_2)$, where $\mathcal{P}(x_3 + C_2)$ is the Weierstrass elliptic function with invariants $g_1 = 0$ and $g_3 = C_1$. Then, according to Sec. 2.3, we obtain the following family of solutions of Eq. (20):

$$u = \mathcal{P}(x_3 + h(\omega)).$$

Here, ω is an arbitrary solution of system (15) and $h(\omega)$ is an arbitrary twice-differentiable function.

Consider the particular case of Eq. (17) where $k = 3$. The ansatz $u = \varphi(\omega)$, $\omega = \omega(x_1^2 + x_2^2 + x_3^2 - x_0^2)^{1/2}$ reduces Eq. (17) to the equation

$$\frac{d^2\varphi}{d\omega^2} + \frac{3}{\omega}\frac{d\varphi}{d\omega} + \lambda\varphi^3 = 0. \tag{21}$$

The exact solutions of Eq. (21) are constructed in [18]. We consider, e.g., the following family of solutions of Eq. (21):

$$\varphi = \frac{1}{a\omega}\tan\left(\pm\frac{\sqrt{2}}{a^2}\ln(C_1\omega)\right), \qquad \lambda = -a^2 < 0. \tag{22}$$

If C_1 is an arbitrary twice-differentiable function $h_1(\omega_1)$, where ω_1 is an arbitrary solution of the system

$$\Box\omega_1 = 0, \qquad (\nabla\omega_1)^2 = 0,$$

$$\nabla\omega\cdot\nabla\omega_1 = 0,$$

then we obtain the following family of solutions of Eq. (17):

$$u = \frac{1}{a\omega}\tan\left(\pm\frac{\sqrt{2}}{a^2}\ln(h_1(\omega_1)\omega)\right).$$

One more family of exact solutions of Eq. (17) with $k = 3$ can be obtained as follows: Applying the group transformation to solution (22), we obtain the family of solutions

$$u = \frac{1}{a}[x_1^2 + x_2^2 + (x_3 + C_1)^2 - x_0^2]^{-1}\tan\left(\pm\frac{\sqrt{2}}{a^2}\ln C_2[x_1^2 + x_2^2 + (x_3 + C_1)^2 - x_0^2]\right).$$

Replacing the constants C_1 and C_2 by arbitrary functions $h_1(\omega_1)$ and $h_2(\omega_1)$, we obtain a more general family of solutions of Eq. (17).

Consider the d'Alembert equation

$$\frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} - \frac{\partial^2 u}{\partial x_3^2} - \frac{\partial^2 u}{\partial x_4^2} + \lambda u^k = 0, \qquad k \neq 1, \tag{23}$$

in the pseudo-Euclidean space $R_{2,2}$. The symmetry ansatz $u = \varphi(\omega_1, \omega_2)$, $\omega_1 = x_1 - x_4$, $\omega_2 = x_1^2 + x_2^2 - x_3^2 - x_4^2$, reduces Eq. (23) to the equation

$$4\omega_1\varphi_{12} + 4\omega_2\varphi_{22} + 8\varphi_2 + \lambda\varphi^k = 0. \tag{24}$$

The generalized ansatz has the form $u = \varphi(\omega_1, \omega_2, \omega_3)$, where ω_3 is an arbitrary solution of the system

$$\Box\omega_3 = 0, \quad \nabla\omega_3\cdot\nabla\omega_1 = 0, \quad \nabla\omega_3\cdot\nabla\omega_2 = 0. \tag{25}$$

System (25), in particular, has the solution $\omega_3 = (x_1 - x_4)(x_2 - x_3)^{-1}$ and, therefore, the ansatz $u = \varphi(\omega_1, \omega_2, \omega_3)$ reduces Eq. (23) to Eq. (24). Equation (24) was studied in detail in [19], where the invariance algebra of this equation in the Lie sense was determined and some classes of exact solutions were constructed. We use, e.g., the following exact solution of Eq. (24):

$$\varphi^{1-k} = \frac{\lambda(k-1)^2}{4(k-2)}(\omega_2 + C_1\omega_1)(1 + C_2\omega_1^{k-2}),$$

where C_1 and C_2 are arbitrary constants. Replacing the constants C_1 and C_2 by arbitrary twice-differentiable functions $h_1(\omega_3)$ and $h_2(\omega_3)$, we obtain a broad class of solutions of Eq. (23):

$$u^{1-k} = \frac{\lambda(k-1)^2}{4(k-2)}(x_1^2 + x_2^2 - x_3^2 - x_4^2 + h_1(\omega_3))(1 + h_2(\omega_3)(x_1 - x_4)^{k-2}).$$

4. Exact Solutions of the Liouville and Sine-Gordon Equations

Consider the Liouville equation

$$\Box u + \lambda \exp u = 0. \tag{26}$$

The symmetry ansatz $u = \varphi(\omega_1)$, $\omega_1 = x_3$, reduces Eq. (26) to the equation

$$\frac{d^2\varphi}{d\omega^2} = \lambda \exp \varphi(\omega_1).$$

Integrating the last equation, we establish that φ coincides with one of the following functions:

$$\ln\left\{\left(-\frac{C_1}{2\lambda}\right)\sec^2\left[\frac{\sqrt{-C_1}}{2}(\omega_1 + C_2)\right]\right\}, \quad C_1 < 0, \quad \lambda > 0, \quad C_2 \in R,$$

$$\ln\left\{\frac{2C_1C_2\exp(\sqrt{C_1}\,\omega_1)}{\lambda[1 - C_2\exp(\sqrt{C_1}\,\omega_1)]}\right\}, \quad C_1 > 0, \quad \lambda C_2 > 0,$$

$$-\ln\left(\sqrt{\frac{\lambda}{2}}\,\omega_1 + C\right)^2.$$

Therefore, according to Sec. 2.3, we obtain the following family of solutions of Eq. (26):

$$u = \ln\left\{\left(-\frac{h_1(\omega)}{2\lambda}\right)\sec^2\left[\frac{\sqrt{-h_1(\omega)}}{2}(x_3 + h_2(\omega))\right]\right\}, \quad h_1(\omega) < 0, \quad \lambda > 0,$$

$$u = \ln\left\{\frac{2h_1(\omega)\,h_2(\omega)\exp(\sqrt{h_1(\omega)}\,x_3)}{\lambda[1 - h_2(\omega)\exp(\sqrt{h_1(\omega)}\,x_3)]}\right\}, \quad h_1(\omega) > 0, \quad \lambda h_2(\omega) > 0,$$

$$u = -\ln\left(\sqrt{\frac{\lambda}{2}}\,x_3 + h(\omega)\right)^2,$$

where $h_1(\omega)$, $h_2(\omega)$, and $h(\omega)$ are arbitrary twice-differentiable functions and ω is an arbitrary solution of system (15).

By using, e.g., the solution [12] of the Liouville equation (26)

$$u = \ln \frac{2(s-2)}{\lambda\left[x_0^2 - x_1^2 - \ldots - x_s^2\right]}, \qquad s \neq 2,$$

we find the following broad class of solutions of this equation:

$$u = \ln \frac{2(s-2)}{\lambda\left[x_0^2 - x_1^2 - \ldots - x_L^2 - (x_{L+1} + h_{L+1}(\omega))^2 - \ldots - (x_s + h_s(\omega))^2\right]},$$

where ω is an arbitrary solution of system (16) and $h_{L+1}(\omega), \ldots, h_s(\omega)$ are arbitrary twice-differentiable functions. If $s = 3$, then Eq. (26) has the following family of solutions in the space $R_{1,3}$:

$$u = \ln \frac{2}{\lambda\left[x_0^2 - x_1^2 - x_2^2 - (x_3 + h_3(\omega))^2\right]}.$$

By analogy, for the sine-Gordon equation $\Box u + \sin u = 0$, we obtain the solutions

$$u = 4 \arctan h_1(\omega)\, e^{\varepsilon_0 x_3} - \frac{1}{2}(1-\varepsilon)\pi, \qquad \varepsilon_0 = \pm 1, \quad \varepsilon = \pm 1,$$

$$u = 2 \arccos\left[dn(x_3 + h_1(\omega), m)\right] + \frac{1}{2}(1+\varepsilon)\pi, \qquad 0 \leq m \leq 1,$$

$$u = 2 \arccos\left[cn\left(\frac{x_3 + h_1(\omega)}{m}, m\right)\right] + \frac{1}{2}(1+\varepsilon)\pi, \qquad 0 \leq m \leq 1,$$

where $h_1(\omega)$ is an arbitrary twice-differentiable function and ω is an arbitrary solution of system (15).

5. Exact Solutions of the Eikonal Equation

We consider the eikonal equation

$$\left(\frac{\partial u}{\partial x_0}\right)^2 - \left(\frac{\partial u}{\partial x_1}\right)^2 - \left(\frac{\partial u}{\partial x_2}\right)^2 - \left(\frac{\partial u}{\partial x_3}\right)^2 = 1. \tag{27}$$

The symmetry ansatz $u = \varphi(\omega_1)$, $\omega_1 = x_0^2 - x_1^2 - x_2^2 - x_3^2$, reduces Eq. (27) to the equation

$$4\omega_1\left(\frac{\partial \varphi}{\partial \omega_1}\right)^2 - 1 = 0. \tag{28}$$

We seek a generalized ansatz in the form $u = \varphi(\omega_1, \omega_2)$. This ansatz reduces Eq. (27) to the equation

$$4\omega_1\left(\frac{\partial \varphi}{\partial \omega_1}\right)^2 + 2(\nabla\omega_1 \cdot \nabla\omega_2)\frac{\partial \varphi}{\partial \omega_1} + (\nabla\omega_2)^2\left(\frac{\partial \varphi}{\partial \omega_2}\right)^2 = 1. \tag{29}$$

Assume that Eq. (29) coincides with the reduced equation (28). It is obvious that this condition is satisfied if ω_2 is such that

$$(\nabla\omega_2)^2 = 0, \qquad \nabla\omega_1 \cdot \nabla\omega_2 = 0. \tag{30}$$

Solving system (30), we determine the explicit form of the variable ω_2. It is easy to see that an arbitrary function of solution (30) is also a solution of system (30). Integrating Eq. (28), we get $(u + C)^2 = x_0^2 - x_1^2 - x_2^2 - x_3^2$, where C is an arbitrary constant. One can obtain a more general family of solutions of the eikonal equation if C is regarded as an arbitrary solution of system (30).

The symmetry ansatz $u = \varphi(\omega_1, \omega_2)$, $\omega_1 = x_0^2 - x_1^2 - x_2^2$, $\omega_2 = x_3$, is generalized as follows: Let ω_3 be an arbitrary solution of the system of equations

$$\left(\frac{\partial\omega_3}{\partial x_0}\right)^2 - \left(\frac{\partial\omega_3}{\partial x_1}\right)^2 - \left(\frac{\partial\omega_3}{\partial x_2}\right)^2 = 0, \tag{31}$$

$$x_0\frac{\partial\omega_3}{\partial x_0} + x_1\frac{\partial\omega_3}{\partial x_1} + x_2\frac{\partial\omega_3}{\partial x_2} = 0.$$

Then the ansatz $u = \varphi(\omega_1, \omega_2, \omega_3)$ reduces the eikonal equation to the equation

$$4\omega_1\left(\frac{\partial\varphi}{\partial\omega_1}\right)^2 - \left(\frac{\partial\varphi}{\partial\omega_2}\right)^2 = 1. \tag{32}$$

Equation (32) has the solutions [12]

$$\varphi = \frac{C_1^2+1}{2C_1}\left(x_0^2 - x_1^2 - x_2^2\right)^{1/2} + \frac{C_1^2-1}{2C_1}x_3 + C_2,$$

$$(\varphi + C_2)^2 = x_0^2 - x_1^2 - x_2^2 - (x_3 + C_1)^2,$$

which can easily be found by applying the method of symmetry reduction to Eq. (32), which reduces it to ordinary differential equations. Replacing the arbitrary constants C_1 and C_2 by arbitrary functions $h_1(\omega_3)$ and $h_2(\omega_3)$, we obtain the following broader classes of exact solutions of the eikonal equation:

$$u = \frac{h_1(\omega_3)^2+1}{2h_1(\omega_3)}\left(x_0^2 - x_1^2 - x_2^2\right)^{1/2} + \frac{h_1(\omega_3)^2-1}{2h_1(\omega_3)}x_3 + h_2(\omega_3),$$

$$(u + h_2(\omega_3))^2 = x_0^2 - x_1^2 - x_2^2 - (x_3 + h_1(\omega_3))^2.$$

Note that we have thus also derived broad classes of exact solutions of the Born–Infeld equation because it is a differential consequence of the eikonal equation [3].

Consider the following eikonal equation:

$$\left(\frac{\partial u}{\partial x_0}\right)^2 - \left(\frac{\partial u}{\partial x_1}\right)^2 - \left(\frac{\partial u}{\partial x_2}\right)^2 - \left(\frac{\partial u}{\partial x_3}\right)^2 = -1. \tag{33}$$

The symmetry ansatz $u = \varphi(\omega_1)$, $\omega_1 = x_3$, reduces Eq. (33) to the equation $\dot{\varphi}^2 = 1$. The reduced equation has a solution $\varphi = \varepsilon x_3 + C$, where $\varepsilon = \pm 1$ and C is an arbitrary constant. Replacing the constant C by an arbitrary function $h(\omega_2)$, where ω_2 is an arbitrary solution of the equation

$$\left(\frac{\partial u}{\partial x_0}\right)^2 - \left(\frac{\partial u}{\partial x_1}\right)^2 - \left(\frac{\partial u}{\partial x_2}\right)^2 = 0,$$

we obtain a more general family of solutions of Eq. (33):

$$u = \varepsilon x_3 + h(\omega_2).$$

6. On Exact Solutions of a Schrödinger-Type Equation

Consider the equation

$$i\frac{\partial \psi}{\partial t} = k\Box\psi + \psi F(|\psi|), \tag{34}$$

where $\psi = \psi(t, x_0, \ldots, x_n)$ and

$$\Box\psi = \frac{\partial^2\psi}{\partial x_0^2} - \frac{\partial^2\psi}{\partial x_1^2} - \ldots - \frac{\partial^2\psi}{\partial x_n^2}.$$

The symmetry ansatz $\psi = \varphi(t)$ reduces Eq. (34) to the equation

$$i\dot{\varphi} - \varphi F(|\varphi|) = 0. \tag{35}$$

This ansatz is a particular case of a more general ansatz

$$\psi = \varphi(t, \omega), \tag{36}$$

where ω is an arbitrary solution of the system of equations

$$i\frac{\partial \omega}{\partial t} = k\Box\psi, \tag{37}$$

$$(\nabla\omega)^2 = 0).$$

Thus, formula (36) defines a family of solutions of the nonlinear multidimensional equation (34) if φ satisfies (35) and ω is a solution of system (37).

The formula

$$\psi = \exp\left\{-\frac{i\left(x_0^1 - x_1^2 - \ldots - x_n^2\right)}{4kt}\right\}\varphi(\omega_1, \omega_2) \tag{38}$$

is an ansatz for Eq. (34) if $\omega_1 = t$, ω_2 satisfies Eq. (37), and

$$x_0\frac{\partial\omega_2}{\partial x_0}+x_1\frac{\partial\omega_2}{\partial x_1}+\ldots+x_n\frac{\partial\omega_2}{\partial x_n} = 0. \qquad (39)$$

The reduced equation has the form

$$i\frac{\partial\varphi}{\partial t}+\frac{(n+1)i}{2t}\varphi-\varphi F(|\varphi|) = 0. \qquad (40)$$

Therefore, solving Eq. (40) and system (37), (39) and inserting the solutions obtained in formula (38), we get broad classes of exact solutions of Eq. (34).

7. System of Nonlinear d'Alembert and Eikonal Wave Equations

Consider the system of equations

$$\Box u = F(u),$$

$$(\nabla u)^2 = -1, \qquad (41)$$

where $u = u(x_0, x_1, x_2, x_3)$. System (41) was investigated in [14, 20], where it was proved, in particular, that this system is compatible if and only if $F(u) = N(u+C)^{-1}$, where C is an arbitrary constant and T is a discrete parameter that may take one of the values 0, 1, 2, 3. Let us construct broad classes of exact solutions of system (41) in the form $F(u) = 0$ and $F(u) = 3u^{-1}$. It is obvious that the system

$$\Box u = 0, \quad (\nabla u)^2 = -1 \qquad (42)$$

has an invariant solution $u = \varepsilon x_3 + C_1$, where C_1 is an arbitrary constant. Therefore, according to Sec. 2.4, system (42) has the family of exact solutions $u = \varepsilon x_3 + h_1(\omega)$, where the function $\omega = \omega(x_0, x_1, x_2)$ is an arbitrary solution of the system $\Box\omega = 0$, $(\nabla\omega)^2 = 0$. The system

$$\Box u = 3u^{-1}, \quad (\nabla u)^2 = -1 \qquad (43)$$

has the invariant solution $u^2 = x_0^2 - x_1^2 - x_2^2 - x_3^2$. Applying the group transformation to this solution, we obtain the solution $u^2 = x_0^2 - x_1^2 - x_2^2 - (x_3+C)^2$. Thus, according to Sec. 2.4, system (43) has the family of solutions

$$u^2 = x_0^2 - x_1^2 - x_2^2 - (x_3 + h(\omega))^2,$$

where the function $\omega = \omega(x_0, x_1, x_2)$ is a solution of the system

$$\Box\omega_3 = 0, \quad (\nabla\omega)^2 = 0,$$

$$x_0\frac{\partial\omega}{\partial x_0}+x_1\frac{\partial\omega}{\partial x_1}+x_2\frac{\partial\omega}{\partial x_2} = 0.$$

The generalization of these results to an arbitrary number n is quite obvious.

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