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Equilibrium of Elastic Media with Internal NonFlat Cracks

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Abstract

A class of mixed boundary problems on equilibrium of three-dimensional bodies weakened by nonflat cracks on parts of the second-degree surfaces is examined. The general approach to these problems is developed. The solution of Lame vector equation of equilibrium is presented in the form of eigenfunction expansions. The unknown coefficients are found from boundary conditions transferred to a crack surface according to the superposition principle. The principle of displacements and stresses fields continuity is used out of the crack. The result is a coupled system of dual series equations or integral equations. Data obtained are applicable to the study of material damage.

KEY WORDS: *internal nonflat crack, mathematical cut, eigenfunction expansions, ellipsoidal surface, stress-intensity factors*

1. Introduction

In most cases the catastrophic failure of constructions is caused by some hidden internal defects. The scientific and engineering experience proves that cracks in materials are the first link in a sequence of processes leading to their fracture. The internal cracks, in the form of material solidity breaks have been examined by scientific literature for quite a long time. The problems on stress-strain state of elastic finite and infinite bodies with flat cracks of penny-shaped or elliptic forms, are among the best researched ones [1]. However, according to experimental analysis of the surfaces of damaged parts, the initial surfaces of the material breaks were of spherical or ellipsoidal shape, that is they were nonflat. To evaluate the strength of material with internal cracks, one can start with the solution of a class of problems within the elasticity theory for three-dimensional bodies weakened by cracks with curved surface. Such cracks could be modeled by cuts on a part of some surface of revolution with its nonzero curvature. In this case, there is a possibility to vary geometrical parameters of the surface and, by doing this, to bring them closer to the geometry of real cracks.

This paper presents a general approach to solving of mixed boundary problems on equilibrium of three-dimensional bodies weakened by mathematical cuts on parts of the second-degree surfaces. As an example the stress strain state of elastic spatial solid with a crack on a part of oblate ellipsoid under torsional forces is examined. This problem is linked to the study of high-strength composite materials with low-percentage content of the ellipsoidal dispersed particles.

2. Statement of the problem

Let us consider a bounded elastic body with internal crack located on an arbitrary surface S. The crack is assumed to be rather far from the body surface. So, the influence of the body boundary on its stress state near the crack is not significant and finite sizes of the body can be superseded with infinite ones.

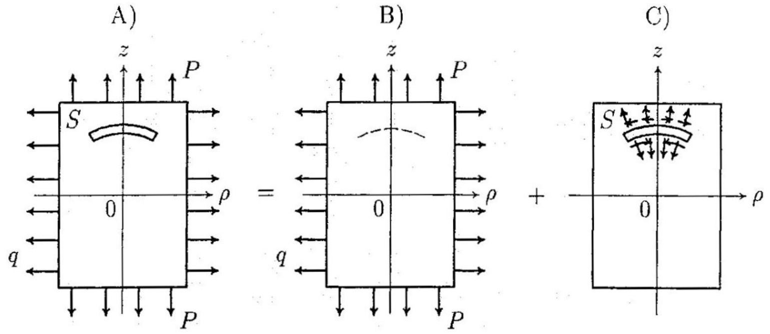

Fig. 1 A superposition principle: the stress field outside the cut (problem A) is a sum of a stress state for a space without the crack (problem B) and a local stress state of the crack surfaces under the load of opposite sign (problem C)

To solve this problem, let us use the superposition principle (Fig.1). Let us search the solution of the problem A as a sum of the solutions of the two following static problems: the problem B on the stress field of the entire elastic body affected by forces and the problem C on the equilibrium of elastic space with a cut on the surface S, when the forces are applied only to the surface of the crack. In the problem B the system of external forces coincides with external forces of the problem A, while in problem C the forces on the surface of the cut are equal in magnitude and opposite in direction to those forces that occur on the conventionally chosen surface of the crack S in the problem B. The superposition of the solution of problems B and C will give a solution to the problem on the equilibrium of the elastic surface with the cut, provided the surfaces of the crack are stressless and the body is influenced by the external field of the axisymmetric forces only. In problems A, B, C the body must have the same geometry and the same mechanical characteristics. This body may be both homogeneous and inhomogeneous. The solution to problem B is relatively simple and the stress field do not have the singularities in the internal points. The character of the singularities of the stresses in the initial problem will result from the solution of problem C. Further research will focus on the problem C with the given forces on the crack's surface [2, 3].

The next stage of the statement of the problem is based on the use of the partial domains method: the complex domain is divided into two or more simpler domains in such a way that the physical fields in each of them will be in the form of eigenfunction expansions. The conditions of continuity of physical fields on the common sections of the neighboring domains have to take place.

Let us divide elastic space into two domains: the inner domain V_1 and external one V_2. For each domain the Lame vector equation of equilibrium

$$2\frac{m-1}{m-2}\,graddiv\,\vec{u}-rotrot\,\vec{u}=0 \tag{1}$$

must be solved. Here $\vec{u}$ is the displacement vector, m is Poisson's number ($m = 1/\nu$, ν is Poisson's ratio). The unknown coefficients are found from the boundary conditions

$$2G\left[\vec{n}\frac{\operatorname{div}\vec{u}}{m-2}+(\vec{n}\cdot\operatorname{grad})\vec{u}+\frac{1}{2}\vec{n}\times\operatorname{rot}\vec{u}\right]_S=\vec{F}_n \tag{2}$$

where G is a shear module of material, $\vec{n}$ is an outward normal to the boundary surface S of the body, $\vec{F}_n$ is a vector of external forces on the surface S.

3. Method of solution

Solutions to the problems on equilibrium of elastic bodies weakened by cuts on the second-degree surfaces of revolution are based on the general solutions to the boundary-value problems of elasticity theory for bodies of revolution in curvilinear coordinates. This paper presents a general approach to solve such a class of problems. The solutions of the Lame vector equation (1) are represented by eigenfunction expansions [3]. After satisfying the boundary conditions (2), we reduce the problems to coupled systems of dual series equations or integral equations in terms of continious or discrete spectrum eigenfunctions. The solutions to these systems will be in the form of special designed integral operators. It allows to reduce all the problems to the following system of integro-diifferential equations

$$\frac{d\varphi_i(x)}{dx}+a_i\varphi_i(x)=\int_0^1[\varphi_i(t)K_{i1}(t,x)+\varphi_2(t)K_{i2}(t,x)]dt=f_i(x)$$
$$b_i\varphi_i(c)+\int_0^1\varphi_i(t)\psi_i(t)dt=d_i \tag{3}$$

where c, a_i, b_i, d_i are the known constants, $f_i(x)$, $\psi_i(x)$ $(i = 1, 2)$ are the given functions for $x\epsilon[0; 1]$; $\varphi_i(x)$ are the unknown functions.The kernels of the system have a break along the diagonal $x = t$. They are represented by formulas

$$K_{ij}(t,x)=K_{ij}^{+}(t,x)H(x-t)+K_{ij}^{-}(t,x)H(t-x) \tag{4}$$

where $H(x-t)$ is the Heavyside function. The last two equations of the system (3) correspond to additional physical or mathematical conditions for the functions $\varphi_i(x)$. Solutions of the system (3) will be in the form of Lagrange interpolating polynomials [4]. The problem on the spherical crack can be solved exactly [3, 5].

4. Torsion of an elastic space with a crack on a part of oblate ellipsoid surface

We consider an elastic medium with a crack in the shape of an ellipsoidal cap ($\xi=\xi_0,\,0\le\eta\le\eta_0,\,0\le\varphi\le2\pi$) in the field of torsional forces symmetric about the axis Oz (Fig.2). Using superscripts (1) and (2) for inner and external domains, respectively, we have the following representations for the displacements and stresses [6]

$$2Gu_1(\xi,\eta)=\sum_{n=1}^{\infty}A_nP_n^1(M)P_n^1(\cos\eta),\ 2Gu_2(\xi,\eta)=\sum_{n=1}^{\infty}B_nQ_n^1(M)P_n^1(\cos\eta),\ M=i\,sh\xi \tag{5}$$

$$h\sigma_{\xi\varphi}^{(1)}=\sum_{n=2}^{\infty}A_nP_n^2(M)P_n^1(\cos\eta),\ h\sigma_{\eta\varphi}^{(1)}=\sum_{n=2}^{\infty}A_nP_n^1(M)P_n^2(\cos\eta) \tag{6}$$

$$h\sigma_{\xi\varphi}^{(2)}=\sum_{n=2}^{\infty}B_nQ_n^2(M)P_n^1(\cos\eta),\ h\sigma_{\xi\varphi}^{(2)}=\sum_{n=2}^{\infty}B_nQ_n^2(M)P_n^1(\cos\eta) \tag{7}$$

where the dimensionless coefficients A_n, B_n are to be determined by applying the boundary conditions on $\xi=\xi_0$:

$$u_1=u_2,\ \sigma_{\xi\varphi}^{(1)}=\sigma_{\xi\varphi}^{(2)},\ (\xi=\xi_0,\ \eta>\eta_0),\ h\sigma_{\xi\varphi}^{(1)}=h\sigma_{\xi\varphi}^{(2)}=f(\eta),\ (\xi=\xi_0,\ 0\le\eta<\eta_0) \tag{8}$$

Satisfaction of these conditions gives the system of dual equations

$$\sum_{n=2}^{\infty}A_nP_n^2(M_0)P_n^1(\cos\eta)=f(\eta);\ (0\le\eta<\eta_0) \tag{9}$$

$$-\frac{i}{\operatorname{ch}\xi_0}\sum_{n=1}^{\infty}\frac{A_nn(n+1)}{Q_n^2(M_0)}P_n^1(\cos\eta)=0;\ (\eta_0\le\eta\le\pi)$$

Here $f(\eta)$ corresponds to loading transferred to the surface of the cut according to the superposition principle. The solution to the system (9) will be with the help of integral operator

$$A_n=2i\operatorname{ch}\xi_0\frac{Q_n^2(M_0)}{n^2(n+1)^2}(n+1/2)\int_0^{\eta_0}\varphi(t)\cos(n+1/2)t\,dt$$

where the unknown auxiliary function $\varphi(t)$ and it derivative are supposed to be continuous for $0\le t\le\eta_0$. The system of dual equations (9) can be reduced to the Fredholm integro-differential equation with respect to the function $\varphi(t)$:

$$\varphi'(x)+a_1\int_0^x\varphi(t)dt+\int_0^{\eta_0}\varphi(t)K(t,x)dt=F(x) \tag{10}$$

$$F(x)=\frac{2}{\pi\sin x}\frac{d}{dx}\int_0^x\frac{f(\eta)\sin^2\eta d\eta}{\sqrt{2\cos\eta-2\cos x}}$$

The kernel $K(t,x)$ in the equation (10) has a form

$$K(t,x)=\frac{1}{2}a_2[2\pi x-(t^2+x^2)H(x-t)-2txH(t-x)]+\frac{4\operatorname{ch}\xi_0}{\pi i}\sum_{n=2}^{\infty}\left[\frac{P_n^2(M_0)Q_n^2(M_0)}{n(n+1)}-\frac{ai}{2\operatorname{ch}\xi_0}\right]\cos\alpha_nt\sin\alpha_nx$$

The integral condition $\int_0^{\eta_0}\varphi(t)\cos\frac{t}{2}dt=0$ completes the equation (10).

The stress component $\sigma_{\xi\varphi}$ on the ellipsoid surface out of the crack, the stress intensity factor K_3 and the crack opening displacement u_2-u_1 are determined by the formulas

$$h_0\sigma_{\xi\varphi}=\frac{-\varphi(\eta_0)\sin\eta_0}{\sin\eta\sqrt{2\cos\eta_0-2\cos\eta}}+\int_0^{\eta_0}\frac{\sin xF(x)dx}{\sin\eta\sqrt{2\cos x-2\cos\eta}}+\int_0^{\eta}\frac{\sin xdx}{\sin\eta\sqrt{2\cos x-2\cos\eta}}\left[\int_0^{\eta_0}\varphi(t)K(t,x)dt\right] \tag{11}$$

$$(h_0=c\sqrt{\operatorname{ch}^2\xi_0-\sin^2\eta})$$

$$K_3=-\varphi(\eta_0)(c\sin\eta_0)^{-0,5}\left(\sqrt{\operatorname{ch}^2\xi_0-\sin^2\eta_0}\right)^{-0,25},\ G(u_1-u_2)=2\int_{\eta}^{\eta_0}\frac{\varphi(t)\sin tdt}{\sin\eta\sqrt{2\cos\eta-2\cos t}} \tag{12}$$

We can write the asymptotic expressions for displacement and stresses near the edge of the ellipsoidal crack:

$$2Gu_2(\xi,\eta) \approx K_3\sqrt{2r}\sin\frac{\gamma}{2};\ \sigma^{(2)}_{\xi\varphi} \approx \frac{K_3}{\sqrt{2r}}\cos\frac{\gamma}{2},\ \sigma^{(2)}_{\eta\varphi} \approx -\frac{K_3}{\sqrt{2r}}\sin\frac{\gamma}{2}.$$

The similar formulas take place in the inner domain.

Let us consider a case when the elastic space with the ellipsoidal cut is in the uniform field of torsional forces. The boundary conditions (8) have the following form $\sigma_{\xi\varphi} = G\theta c^2 \sin 2\eta \operatorname{ch}^2 \xi_0 / 2h_\xi$, where θ is an angle of twisting per unit of length. The right part of the integro-differential equation (10) is equal to $F(x) = \frac{8}{5\pi} G\theta c^2 \operatorname{ch}^2 \xi_0 \sin\frac{5x}{2}$. Fig. 3 shows the behaviour of the dimensionless stress intensity factor $\overline{K}_3 = -\pi K_3 / G\theta$ for $\operatorname{th}\xi_0 = 1; 0.75; 0.5; 0.25$. The parameter c is matched to retain the largest axis of ellipsoid $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$ equal to constant $b = c\operatorname{ch}\xi_0 = \operatorname{ch}3$.

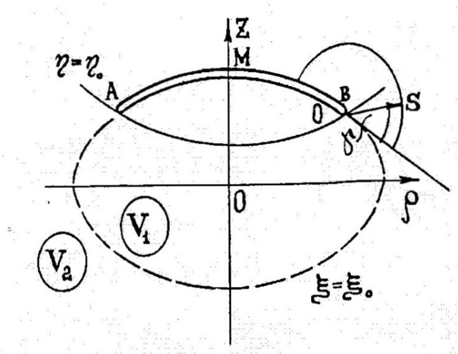

Fig. 2 The ellipsoidal crack in an elastic space. AMB is a meridian section of the crack

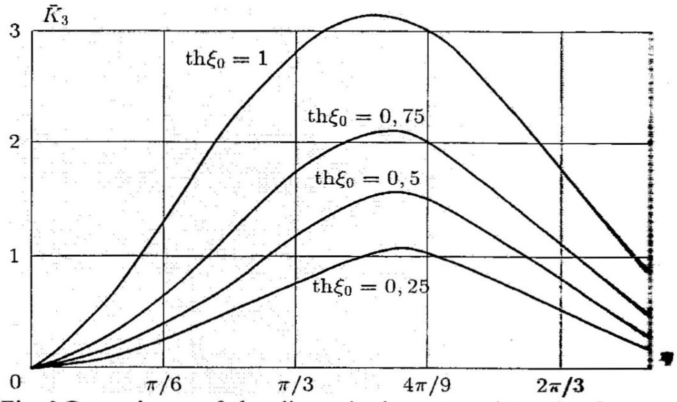

Fig. 3 Dependence of the dimensionless stress-intensity factor on the ellipsoidal crack geometry

5. Conclusions

The obtained analytical solution gives a general picture of mechanical conditions of the system dependency on the changes in the problem's parameters, such as external loading, geometry of the crack, elastic constants, etc., and, thus, allows to foresee the cut's behaviour when these parameters change. The results of this work show the advantages of such approach. In particular, we have found the analytical expressions for the components of stress tensor, crack opening displacement and stress intensity factors (SIF) near the edge of the ellipsoidal crack in an elastic space. In the case when the elastic space with the ellipsoidal cut is in the uniform field of torsional forces the dependence of the dimensionless stress-intensity factor on the ellipsoidal crack geometry is obtained.

References

1. **Liebowitz, H. (Ed.).** Fracture: An Advanced Treatise. Vol.2. Mathematical Fundamentals. - London: Academic Press, 1968.-434p.
2. **Muskhelishvili, N. I.** Some Basic Problems in the Theory of Elasticity.- Noordhoff: The Netherlands, 1966. -707p.
3. **Ulitko, A. F.,** Vector Expansions in the Three-Dimensional Theory of Elasticity. –Kyiv: Akademperiodika,. 2002. -341p. (in Russian).
4. **Kantorovich, L. V., Krylov, V. N.** Approximated Methods of Higher Analysis. –Moscow –Leningrad: Fizmatgiz, 1962. -708p. (in Russian).
5. **Martin, P. A.** The spherical-cap crack revisited. -Int. J. Solids and Structures, 2001,Vol. 38, p.4759-4776.
6. **Martynenko, M. A., Lebedyeva, I. V.** The torsion of elastic space weakened by the crack along a part of oblate ellipsoid surface. –Visnyk of Kyiv Taras Shevchenko National University, mathematics, mechanics, 2005, No13-14, p.62-66. (in Ukrainian).