EXACT SOLUTIONS OF AN EQUATION OF GAS DYNAMICS

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A new wide class of exact solutions of the gas dynamics equation $\Box u + \lambda u u_0 = 0$ is obtained.

The equation

$$\Box u + \lambda u u_0 = 0, \tag{1}$$

where

$$\Box u = \frac{\partial^2 u}{\partial x_0^2} - \frac{\partial^u}{\partial x_1^2} - \ldots - \frac{\partial^2 u}{\partial x_n^2},$$
$$u_0 = \frac{\partial u}{\partial x_0},$$

λ is an arbitrary real number different from zero, was the subject of investigation in papers [1-4]. The equation (1) occurs in the theory of field and gas dynamics. It is shown in [1] that the maximal invariance algebra of equation (1) in Lie's sense is an algebra F generated by the vector fields

$$P_0 = \frac{\partial}{\partial x_0},$$
$$P_a = \frac{\partial}{\partial x_a}, \quad J_{ab} = x_a \frac{\partial}{\partial x_b} - x_b \frac{\partial}{\partial x_a},$$
$$D = x_0 \frac{\partial}{\partial x_0} + x_1 \frac{\partial}{\partial x_1} + \ldots + x_n \frac{\partial}{\partial x_n} - u \frac{\partial}{\partial u},$$

where $a, b = 1, 2, \ldots, n$.

Some exact solutions of the two-dimensional equation (1) for the case $\lambda = 2$ were found in [2] and of three- and four-dimensional equations in [1,3]. In [4], the partial solutions of equation (1) were constructed for the four-dimensional case by the use of symmetry reduction of equation (1) to the ordinary differential equations,

In the present paper, a new wide class of exact solutions to equation (1), which depends on two arbitrary harmonic functions, is constructed. It should be noted that these solutions are invariant under no subalgebra of the algebra F and, for this reason, they are not Lie's solutions. For constructing these solutions, we don't employ the notion of the Lie invariance of equation (1) [3].

1. Let us consider the first case $n = 1$. We shall seek a solution of equation (1) in the form $u = a(x_0)b(x_1)$, where functions $a(x_0)$ and $b(x_1)$ differ from constants. Substituting it into equation (1), we get

$$a''b - ab'' + \lambda a a' b^2 = 0. \tag{2}$$

Here and below, a' and a'' mean, respectively, the first and second derivatives of the function $a(x_0)$ with respect to the variables x_0. It follows from Eq. (2) that functions b, b^2, b'' are linearly dependent. If $b'' = 0$, then $a''b + \lambda a a' b^2 = 0$. Since functions b and b^2 are linearly independent, it follows from the latter equation that $a'' = 0$, $aa' = 0$. From it we find $a = \text{const}$ that contradicts the assumption. Therefore, $b'' \neq 0$ and, hence, b'' is a linear combination of functions b and b^2, i.e., $b'' = \alpha b + \beta b^2$. Substituting it into equation (2), we obtain

$$(a'' - \alpha a)b + (-\beta a + \lambda a a')b^2 = 0,$$

where

$$a'' - \alpha a = 0, \qquad -\beta a + \lambda a a' = 0.$$

The obtained system of equations possesses a trivial solution $a = 0$ if $\alpha \neq 0$. If $\alpha = 0$, then $a'' = 0$, $\lambda a' = \beta$, $b'' = \beta b^2$. Multiplying the function b by a real number $\beta/6$ and the function a by $6/\beta$, we may always take $\beta = 6$. Therefore, we have the following solutions of (1):

$$u = \left(\frac{6}{\lambda} x_0 + \nu\right) \wp(x_1), \quad \wp'' = 6\wp^2, \tag{3}$$

$$u = \left(\frac{6}{\lambda} x_0 + \nu\right) \frac{1}{x_1^2}, \tag{4}$$

where $\wp(x_1)$ is the Weierstrass function with invariants $g_2 = 0$, $g_3 = C_1$; ν is an arbitrary real number.

Solution (3) is a partial case of the more general solution

$$u = \left(\frac{6}{\lambda} x_0 + \nu\right) \wp(x_1) + f(x_1). \tag{5}$$

Substituting it into equation (1), we get $f'' = 6\wp f$, i.e., $f(x_1)$ is the Lamé function. Another generalization of solution (3) is of the form

$$u = \left(\frac{6}{\lambda}x_0 + \nu\right)\wp(x_1) + \frac{12}{\lambda^2}\left(\frac{6}{\lambda}x_0 + \nu\right)^{-1},$$

which may be transferred by a group transformation into the solution

$$u = \frac{6}{\lambda}x_0\wp(x_1) + \frac{2}{\lambda x_0}. \tag{6}$$

Solutions

$$u = \left(\frac{6}{\lambda}x_0 + \nu\right)\frac{1}{x_1^2} + Cx_1^3, \tag{7}$$

$$u = \left(\frac{6}{\lambda}x_0 + \nu\right)\frac{1}{x_1^2} + \frac{2}{\lambda x_0} \tag{8}$$

are generalizations of solution (4).

2. Let us consider the case $n > 1$. If $n = 2$, then the solutions

$$u = \left(\frac{6}{\lambda}x_0 + \alpha x_1 + \beta\right)\wp(x_1) + (\delta x_1 + \sigma)f(x_2) \tag{9}$$

and

$$u = \left(\frac{6}{\lambda}x_0 + \alpha x_1 + \beta\right)\frac{1}{x_2^2} + (\delta x_1 + \sigma)x_2^3, \tag{10}$$

are generalizations of solutions (5) and (7), where $\wp'' = 6\wp^2$, $f'' = 6\wp f$, and $\alpha, \beta, \delta, \sigma$ are arbitrary real numbers.

If $n = 3$, then we obtain the following generalizations of solutions (5) and (7):

$$\begin{aligned} u &= \left(\frac{6}{\lambda}x_0 + G_\alpha(x_1, x_2)\right)\wp(x_3) + \\ &+ \Phi_\alpha(x_1, x_2)f(x_3), \end{aligned} \tag{11}$$

$$\begin{aligned} u &= \left(\frac{6}{\lambda}x_0 + G_\alpha(x_1, x_2)\right)\frac{1}{x_3^2}) + \\ &+ \Phi_\alpha(x_1, x_2)x_3^3, \end{aligned} \tag{12}$$

where $\wp'' = 6\wp^2$, $f'' = 6\wp f$,

$$\frac{\partial^2 G_\alpha}{\partial x_1^2} + \frac{\partial^2 G_\alpha}{\partial x_2^2} = 0, \quad \frac{\partial^2 \Phi_\alpha}{\partial x_1^2} + \frac{\partial^2 \Phi_\alpha}{\partial x_2^2} = 0,$$

i.e., $G_\alpha(x_1, x_2)$ and $\Phi_\alpha(x_1, x_2)$ are arbitrary harmonic functions. A generalization of solutions (5) and (7) for the case $n > 3$ is quite obvious.

3. Solution (7) for the case $n > 1$ can be generalized in the following way. Let us consider a subalgebra $\langle J_{12}, J_{13}, \ldots, J_{k-1,k}, P_{k+1}, \ldots, P_n\rangle$ $(k \geq 1)$ of the invariance algebra F for equation (1). The main invariants of the subalgebra L are functions u, $\omega_0 = x_0$, $\omega_1 = x_1^2 + \ldots + x_k^2$. The ansatz $u = \varphi(\omega_0, \omega_1)$, corresponding to the subalgebra L, reduces equation (1) to the equation

$$\varphi_{00} - 4\omega_1\varphi_{11} - 2\kappa\varphi_1 + \lambda\varphi\varphi_0 = 0, \tag{13}$$

where

$$\begin{aligned} \varphi_0 &= \frac{\partial\varphi}{\partial\omega_0}, \quad \varphi_1 = \frac{\partial\varphi}{\partial\omega_1} \\ \varphi_{00} &= \frac{\partial^2\varphi}{\partial\omega_0^2}, \quad \varphi_{11} = \frac{\partial^2\varphi}{\partial\omega_1^2}. \end{aligned}$$

We seek a solution to equation (13) in the form $\varphi = a(\omega_0)b(\omega_1)$, where functions $a(\omega_0)$ and $b(\omega_1)$ differ from constants. Substituting into equation (13), we come to

$$a''b - 4\omega_1 ab'' - 2\kappa ab' + \lambda aa'b^2 = 0. \tag{14}$$

Equation (14) means that the functions a'', a, aa' are linearly dependent. Let us suppose the functions a, aa' are linearly independent. Then the function a'' is a linear combination of a and aa':

$$a'' = \alpha a + \beta aa', \qquad \alpha, \beta \in \mathbb{R}.$$

Substituting a'' into (14) end equating to zero coefficients at aa', we get $\beta b + \lambda b^2 = 0$, i.e., $b = \text{const}$, that is impossible. The obtained contradiction proves that the functions a and aa' are linearly dependent, therefore, $a' = \mu = \text{const}$. In view of this condition, equation (14) takes the form

$$4\omega_1 b'' - 2\kappa b' - \lambda\mu b^2 = 0. \tag{15}$$

If we find a solution to (15), then the formula $u = a(\omega_0)b(\omega_1)$ will give us a solution to Equation (1). We shall seek a partial solution to Equation (15) in the form $b = t\omega_1^\alpha$ [5]. Substituting it into Equation (15), we come to

$$4\alpha(\alpha - 1)t\omega_1^{\alpha-1} + 2\kappa\alpha t\omega_1^{\alpha-1} - \lambda\mu t^2\omega_1^{2\alpha} = 0. \tag{16}$$

It follows from Equation (16) that $\alpha = -1$. Therefore,

$$8t - 2\kappa t - \lambda\mu t^2 = 0.$$

Solving this equation, we get $t = \dfrac{8 - 2\kappa}{\lambda\mu}$. Hence, we have constructed the solution

$$u = \frac{8 - 2\kappa}{\lambda}\frac{x_0}{x_1^2 + \cdots + x_\kappa^2} \tag{17}$$

of equation (1).

The solution (17) is a partial case of the more general solution

$$u=\frac{8-2\kappa}{\lambda}\frac{x_0}{x_1^2+\cdots+x_\kappa^2}+f(\omega_1).$$

Substituting it into Equation (13), we obtain

$$\omega_1^2 f''+\frac{\kappa}{2}\omega_1 f'-\left(2-\frac{\kappa}{2}\right)f=0. \qquad (18)$$

The Equation (18) is the Euler equation. Its general solution is of the form [5]

$$f=C_1\omega_1^{-1}+C_2\omega_1^{2-\kappa/2}.$$

Hence, equation (1) possesses the solution

$$\begin{aligned} u &= \frac{8-2\kappa}{\lambda}\frac{x_0+C_1}{x_1^2+\cdots+x_\kappa^2}+\\ &+ C_2(x_1^2+\cdots+x_\kappa^2)^{2-\kappa/2}. \qquad (19)\end{aligned}$$

It is easy to see that solution (19) is a partial case of the more general solution

$$\begin{aligned} u &= \qquad (20)\\ &= \frac{8-2\kappa}{\lambda}\frac{x_0+C_\alpha(x_{\kappa+1},\ldots,x_n)}{x_1^2+\cdots+x_\kappa^2}+\\ &+ \Phi_\alpha(x_{\kappa+1},\ldots,x_n)(x_1^2+\cdots+x_\kappa^2)^{2-\kappa/2},\end{aligned}$$

where

$$\frac{\partial^2 G_\alpha}{\partial x_{\kappa+1}^2}+\cdots+\frac{\partial^2 G_\alpha}{\partial x_n^2}=0,$$

$$\frac{\partial^2 \Phi_\alpha}{\partial x_{\kappa+1}^2}+\cdots+\frac{\partial^2 \Phi_\alpha}{\partial x_n^2}=0.$$

If we put $\kappa=1$ in (20), then we obtain solution (7).

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ТОЧНІ РОЗВ'ЯЗКИ РІВНЯННЯ ГАЗОВОЇ ДИНАМІКИ

А.Ф. Баранник, І.І. Юрик

Р е з ю м е

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ТОЧНЫЕ РЕШЕНИЯ УРАВНЕНИЯ ГАЗОВОЙ ДИНАМИКИ

А.Ф. Баранник, И.И. Юрик

Р е з ю м е

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